Asset Pricing with Endogenous Disasters

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We develop a parsimonious model in which frictions in the labor market may turn small, continuous labor productivity declines into large drops in employment, endogenously causing disasters. Assuming one state variable and CRRA agents, we solve for prices in closed form, calibrate the model using labor market data, and show that this simple setting captures the high, countercyclical volatility and equity premium observed in the United States. Moreover, returns in our model are conditionally predicted by dividend yields. Finally, as in the data, in our setting the disasters are larger when the capital's share of income is higher. (*JEL* G12)

Asset pricing models with rare catastrophic events may produce a high equity premium (Rietz 1988; Barro 2006), offering an explanation for the equity premium puzzle (Mehra and Prescott 1985). These models may also be extended to produce high volatility (Wachter 2013), thereby providing a resolution to the volatility puzzle (Shiller 1981; Le Roy and Porter 1981; Keim and Stambaugh 1986; Campbell and Shiller 1988; Hodrick 1992). However, in these studies the cause of the disasters is unspecified, making it difficult to assess the validity of the model other than by observing that it calibrates reasonably (Mehra and Prescott 1988).[1]

The model developed in this paper contributes to the aforementioned catastrophic risk literature. First and foremost, in contrast with previous research, the disasters in our model are endogenous. More precisely, these disasters are generated by labor dynamics, and, as such, the model may elucidate

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[1] Some of the models incorporating disasters are also referred to as "peso models." The reason lies in the collapse of the Mexican peso in 1994. The high peso premium observed prior to the collapse was explained by the small probability of a huge out-of-sample devaluation, which was eventually observed (see Danthine and Donaldson 1999). Other models allow for disasters of magnitudes not yet observed in the data. Modeling large but never observed catastrophes led Mehra and Prescott (1988) to criticize the early disaster models as ad hoc.

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how asset prices and employment are related. Within this setting, agents respond to economic conditions, suggesting that a disaster is near by increasing the volatility of the risky asset and driving up its expected returns. These responses also create conditional patterns in prices, which, as Veronesi (2004) argued, can be tested even if disasters are not observed in the data. Within the context of this parsimonious model, we address the puzzles enumerated above, using a setting in which agents have CRRA utility with reasonable levels of risk aversion.

Our economic setting shares a number of features with Diamond (1982), whose study links the labor market with the output market. The model we study additionally incorporates stock markets and calculates the corresponding asset prices. Our simple economy has two sectors. In the first of these sectors, capital (which is the focus of asset pricing) and labor meet, and the consumption good is generated. The production technology in this sector is of Cobb-Douglas type, but as in Diamond (1982), it is also directly proportional to total employment in this sector. This latter feature alludes to an economy producing specialized, nonhomogenous goods that must be first traded with other employed laborers before they are consumed.[2] Having such an externality in the production function links this sector's output to the labor markets: the output will be higher when labor is committed to this sector rather than being employed somewhere else. We further assume that the production technology is directly proportional to an exogenous productivity variable, which in our model is continuous and does not exhibit any jumps. If and when labor decides to leave the first economic sector, it can move to a second economic sector, which is less capital intensive than the first. For tractability, we assume that the production technology in the second sector requires only labor and that it generates some alternative wage (expressed in units of the consumption good) per unit of labor. For simplicity, and consistent with observations on wages' downward rigidity, we assume this alternative wage to be deterministic, increasing at a constant rate. Finally, laborers and capital owners are price takers and act to maximize their CRRA utility.

As observed by Diamond (1982), this specification produces a fragile economic setting. To see why, suppose that productivity decreases substantially. As this will lower the output, the wages paid to laborers in the capital-intensive sector will decline. If this decline causes wages to drop beyond the alternative wage, the marginal laborer might decide to leave the capital-intensive sector and work for the alternative wage instead. Because output in the capital-intensive sector depends on total employment, this, in turn, will depress wages even more and trigger a further decrease in employment in the capital-intensive sector. This cascading mechanism will unfold throughout our economy, triggering a drop

2 For example, in Diamond (1982), laborers on a tropical island produce only coconuts, and a taboo prevents them from consuming their own pick. To consume, Diamond's laborers need to find a trade partner (while incurring a search cost), and exchange coconuts (at a coconut-for-coconut rate). In this way, while the model has one consumption good, it exhibits similarities to an economy with a variety of goods in which producers need to sell a portion of their output, being unable to consume it all.

in rents on capital, and a drop in overall consumption that intuitively is equal to the capital's share of income. This is the disaster in the sense understood by the Rietz-Barro-like literature. In contrast with previous research, however, this negative jump in consumption occurs endogenously via a well-specified mechanism, whereas our state variable does not have any jumps. To further strengthen the endogenous aspect of the economic disasters in our economy and prevent "sunspots," or self-fulfilling cataclysmic events (Cass and Shell 1983; Azariadis 1981), we assume away the possibility that workers may arbitrarily shift into the less capital-intensive sector.

The dynamics of our model are consistent with several stylized facts of financial crises and their recovery. The model includes the vicious feedback cycle of depressions and other financial crises, in which unemployment begets unemployment: as firms downsize in response to a slowdown in the economy, the workers who become unemployed can afford less, depressing aggregate demand and causing a further slowdown. The model also mirrors the virtuous employment cycle of economic recovery, in which newly hired workers can spend more, pushing up aggregate demand and strengthening recovery and growth. Furthermore, the model also captures the fact that economies are more unstable in bad times. This lack of stability can manifest in multiple ways, ranging from effects as simple as unprecedented policy responses (which, without history as a guide, are obviously risky), to complex changes such as shifts in government, antidemocratic measures, military coups, social uprisings, or revolutions. While our model, being parsimonious, has only one (extreme) bad state, when calibrating we apply the less extreme interpretation of a disaster and a correspondingly milder drop in consumption. However, our model is by no means intended to capture all of the full and true dynamics of crises and recoveries. For example, we abstract away from money and credit, from monetary and fiscal policy (and policy mistakes), and from heterogeneity or time delays in labor, capital, and the consumption good. Instead, we propose a parsimonious model, which allows us to solve atemporal and temporal equilibria with rational agents in a tractable way; this in turn allows us to price long-lived assets in the economy and address a variety of asset pricing puzzles.

In the setting outlined above, the state variable—the productivity—affects asset prices through three concurrent mechanisms. The first mechanism is as in Lucas (1978): a reduction in output reduces prices because the level of expected cash flows is lower. The second mechanism is that lower productivity means that a disaster is more imminent, and hence the duration at which dividends are received is shorter.[3] The third mechanism stems from the fact that through a risk aversion effect, when a disaster is closer, the pricing kernel

[3] This is similar to having probabilities of disaster that are perfectly negatively correlated with consumption, as in Gourio (2008a) or Wachter (2013). In contrast with their work, the probability of disaster in our model is endogenous.

of the model becomes lower. A decline in productivity induces a simultaneous decline in dividends, in the duration that these dividends are being paid, and also in the pricing kernel, thereby not only decreasing prices but doing so substantially. Because a small change in dividends may result, through the coordination of these three mechanisms, in a big change in prices, our model can accommodate a small volatility of dividends with a high volatility of prices, thus addressing the volatility puzzle. Additionally, consumption and equity prices are correlated, and, in turn, because of the high volatility of prices relative to their fundamentals, the equity premium will be high. Finally, our analysis shows that the sensitivity of prices to dividends is stronger when the economy is weaker, leading to countercyclical volatility and equity premium.

We calibrate the model by following Barro (2006) and Mehra and Prescott (1985), assuming an annual volatility of consumption of 3.57% and a relative risk aversion under 10. With these parameters we are able to provide a full resolution to both the excess volatility puzzle and the equity premium puzzle. In addition, given the fact that our disaster-inducing mechanism is fully specified, we are able to check the validity of our model independently of asset pricing. For example, our model produces the simple implication that more capital-intensive economies experience larger disasters. We use cross-sectional country data reported in Gollin (2002) for labor's share of income, and disaster magnitudes from Barro and Ursua (2008) to check this prediction and find support for it. In our model, the drop in consumption when a disaster occurs is determined only by the capital's share of income, and for U.S. data, in particular, we predict this drop to be 36%. This is similar to the calibrations used in Barro (2006). Moreover, in our calibration the duration of disasters, as well as the growth from the trough, appear consistent with those reported, for example, by Gourio (2008b).

As in Veronesi (2004), by using conditional moments of returns, we are able to test our model regardless of whether an actual disaster has been observed in the data. We test whether conditional volatility and returns follow our predictions and find strong support for our model. Specifically, as observed in the data, volatility in our model is countercyclical. Furthermore, the expected excess returns depend on dividend yields in a simple way, and this dependence offers a theoretical justification for why returns should be predicted by dividend yields. In fact, in some cases, the dividend yield implied by our model completely subsumes the predictive power of the observed dividend yield and of another traditional predictor, namely, the *cay* variable of Lettau and Ludvigson (2001). With certain calibrations, our model suggests a theory of why the power of predictive regressions is countercyclical, as documented by Henkel, Martin, and Nardari (2011). Predictability of returns in our model does not make the agents irrational, as it is driven by predictable changes in expected returns.

The paper is organized as follows: Section 1 presents the model of the economy and solves for dividends, consumption, and asset price levels.

Section 2 discusses the testable predictions generated from our model. Section 3 calibrates the model and tests it, and Section 4 offers our conclusions.

1. The Model

In this section we introduce our model, which is that of a two-sector, dynamic production economy. This setting is a simplified version of Matsuyama (1991), which is in turn inspired by Diamond (1982). We start by presenting our model's assumptions.

1.1 The real economy

There is a single perishable consumption good, which also serves as the numeraire.

There is an infinite number of households in the economy, indexed by $j \in \mathcal{J}$, all infinitely lived. Households are endowed with a flow of labor services, according to a measure $L_t^{\mathcal{E}}$, defined on $\mathcal{J}$ with a total mass of one. Households are also endowed with a capital asset K with a total mass of one. In our model, K cannot be accumulated and does not depreciate. There are other securities besides the capital asset (e.g., bonds) that are assumed to be in zero supply.

There are two sectors in the economy. In one sector, there is an infinite number of firms, indexed by $i \in \mathcal{I}$. Firms rent labor services and the use of capital and produce the consumption good. Firms do not own any assets and do not trade in financial assets. Firms can only rent labor services and capital at the prevailing spot prices and sell their output at the spot price of the consumption good (which is also the numeraire). In this first sector of the economy, each firm i uses the same production technology at time t. The production function is modeled to have a Cobb-Douglas form, with an externality originating in Diamond (1982). As outlined in Diamond (2011), the particular functional form employed in the production function allows for an interaction between the labor market and the output market. To create such an interaction, the standard Cobb-Douglas production function is multiplied by a function of labor. Because the employment rate is procyclical,[4] the simplest such function is an affine, nondecreasing one. With these considerations, for each firm i, this production function is given by

$$F_t(L_{it}, K_{it}) = \theta_t \bar{L}_t L_{it}^{1-a} K_{it}^{a} \tag{1}$$

$$\bar{L}_t = \sum_{i \in \mathcal{I}} L_{it},$$

where L_{it} and K_{it}, respectively, are the amounts of labor and capital supplied to firm i at time t. The production function of each firm is directly proportional

[4] See, for example, Barro's (2007) Figure 9.7.

to the amount of labor $\bar{L}_t$ supplied in aggregate to the first sector. Intuitively, when less labor is supplied in aggregate, the trading costs each firm incurs to sell its output are higher, and therefore the usable output generated by each firm is lower.[5] The capital's share of income is a. An economy in which the services sector is preponderant would serve as a good example for understanding the externality in this production function. In such an economy, demand for the production good is naturally lower when unemployment is higher, and in equilibrium production adjusts accordingly.

The variable θ, the sole state variable of our model, is a continuous, positive diffusion. Modeling choices for θ_t will be detailed in Section 1.6.

We further assume that laborers have the choice to work instead in a second economic sector generating an alternative wage $Z_t > 0$. In practice, because the labor's share of income is countercyclical,[6] in bad times (which in our model are the times when productivity, and therefore output, is lower) the sectors to which the labor migrates are less capital intensive. For simplicity, we shall assume that the alternative to the capital intensive sector is one in which the capital share of income is zero; in other words, we assume that the less capital-intensive sector has a production function of the form:

$$F_t(L) = Z_t L. \tag{2}$$

We emphasize that both technologies produce the same type of consumption good, in particular, the goods generated through the less capital-intensive technology are part of the overall consumption.

The interpretation of the alternative production sector deserves some discussion. One may contrast it with the first sector, which is highly specialized and very efficient. The economy in the first sector has thousands of specialized factories, each employing unique, sophisticated machinery and labor skills. The efficiency of the overall economy depends on the ability to get thousands of different products to the consumer with as little friction as possible.[7]

5 The steps to arrive at this functional form of the production functions are as follows. First, we start with a simple Cobb-Douglas production function, $F_t^1(L_{it}, K_{it}) = L_{it}^{1-a} K_{it}^a$. This is the gross output of the firm. However, as in Diamond (1982), each firm needs to find a trade partner (other firms) to whom it can sell its output. This is possible to the extent that other firms are operating. We assume that this search cost is a fraction of the output of the firm that is proportional with total employment in firms. We assume that the search cost is a fraction $(1-\bar{L}_t)$ of the gross output. This results in a net output equal to $F_t^2(L_{it}, K_{it}) = \bar{L}_t L_{it}^{1-a} K_{it}^a$. To this we add an exogenous productivity factor θ_t (modeling the fact that sometimes the coconut trees make more coconuts for the same amount of labor and capital invested in the tree) and thus obtain the production function of Equation (1).

6 For example, Gomme and Greenwood (1995) cite this as a stylized fact about the labor's share of income. They report a correlation of -0.37 between labor's share of income and GNP (both detrended using a Hodrick-Prescott filter).

7 Diamond (2011) states that "[…] production contributes to demand […] when others are producing little, the ability to sell is low, and so the incentive to produce is low. When others are producing much, the incentive to produce is high." Furthermore, analyzing the same relationship between employment and output, in a *Wall Street Journal* interview (on August 12, 2011), economist Nouriel Robini alluded to such an externality, arguing that the less companies hire, the less money flows into the hands of workers, and therefore the less companies sell, and the more they are forced to lay off workers. Additionally, Roubini states that in an attempt to increase

The alternative production sector would still have a diversified output, but it would function less efficiently, reminiscent of how the economy worked before the highly specialized, capital-intensive sector existed. Trade and search are more localized. Crises might be localized, but there is less contagion, and the overall economy is more robust.

A second way to interpret the alternative production sector is as conventional unemployment, or being out of the labor force. The alternative wage, which could be thought of as a reservation wage in this interpretation, could come from unemployment benefits, from adding value at home, or other valuable, but nonpaying, activities. It also can be partly irrational. For example, with the wage rigidity phenomenon (Kramarz 2001; Dickens et al. 2007), in which lower-paying jobs are not filled even when better alternatives are lacking.[8]

Next, we will describe the agents' preferences.

1.2 Preferences

The agents in this economy have time-separable utilities and are not satiated. The utility of the representative agent j from a consumption stream $C=(C_t)_{t\geq 0}$ is given by:

$$U(C)=\mathbb{E}\left[\int_0^\infty e^{-\delta t}u(C_t)dt\right], \tag{3}$$

where δ is a discount rate. In the analysis of the real side of the economy, all we require is that $u(\cdot)$ is a strictly increasing function. We shall use constant relative risk aversion preferences, that is, $u(C)=(1-\gamma)^{-1}C^{1-\gamma}$ with $\gamma>1$.

We continue by describing the households' and firms' optimization problems.

1.3 Households' optimization problem

Each household j is endowed at time t with an amount of labor $L^{\mathcal{E}}_{jt}$ of which it decides to supply $L^{\mathcal{S}}_{jt}$ to the first sector by working for firms. The remainder $L^{\mathcal{E}}_{jt}-L^{\mathcal{S}}_{jt}$ works in the sector that pays the alternative wage. Each household may supply capital and may hold the securities $s\in S_j$, in proportion of π_{jst} (one of these securities may be the capital asset K), and S_j is a subset of S, the set of all securities.

productivity, companies might lay off labor, also causing less money to flow into the hands of workers, thereby decreasing output and forcing even more unemployment. Although we do not model this particular feedback cycle, doing so has the potential to make disasters more frequent and would result in a higher equity premium and volatility.

[8] Note that our interpretation and calibration of the less capital-intensive sector is not one of a primitive economy, although the model itself could be interpreted that way. To calibrate to an extreme disaster state, we would have to assume that the second economic sector uses solely raw labor in the production function and that its output is drastically lower than in our calibration. This would result in much more severe disasters than our current calibration implies and would be beyond the scope of this study.

The total flow of income that household j receives at time t is the sum of wages paid by capital-intensive firms, the output of labor assigned to the less capital-intensive sector, and the dividends paid by the securities it holds:

$$D_{Lt}L_{jt}^{\mathcal{S}}+Z_t\left(L_{jt}^{\mathcal{E}}-L_{jt}^{\mathcal{S}}\right)+\sum_{s\in S_j}D_{st}\pi_{jst}. \tag{4}$$

This income is used for consumption at a rate C_{jt} or to finance rebalancing of the household's portfolio. Therefore, at each time t the budget constraint faced by a household is

$$C_{jt}dt+\sum_{s\in S_j}P_{st}d\pi_{jst}\leq\left[D_{Lt}L_{jt}^{\mathcal{S}}+Z_t\left(L_{jt}^{\mathcal{E}}-L_{jt}^{\mathcal{S}}\right)+\sum_{s\in S_j}D_{st}\pi_{jst}\right]dt. \tag{5}$$

Households are price takers. In that respect, a household observes the current wage D_{Lt} and the prices P_{st}, as well as the dividends generated by ownership in the capital asset D_{Kt}, and maximizes its utility from consumption, as defined in Equation (3), by selecting the amount of labor supplied to the firms $L_{jt}^{\mathcal{S}}$ and by deciding which securities (including the capital asset K) to hold in its portfolio. As the utility u is increasing, the household must first maximize the upper bound on consumption, which it controls, for example, through the labor supply $L_{jt}^{\mathcal{S}}$. The following proposition describes the optimal amount of labor supplied by a household.

Lemma 1. The optimal decision of the representative household may be described as follows. At any time t:

(1) When $D_{Lt}>Z_t$, each household prefers to rent out all its labor endowment to the first sector (i.e., resulting in $\bar{L}_t=1$ in the entire economy).
(2) When $D_{Lt}<Z_t$, each household uses all of its labor endowment in the technology paying the alternative wage (i.e., $L_{jt}=0$ for all j).
(3) When $D_{Lt}=Z_t$, a household may rent out any amount of labor $L_{jt}^{\mathcal{S}}\in\left[0,L_{jt}^{\mathcal{E}}\right]$ to the first sector and use the remainder $L_{jt}^{\mathcal{E}}-L_{jt}^{\mathcal{S}}$ to generate the alternative wage (i.e., any $\bar{L}_t\in[0,1]$ may result as a solution to the households' optimization problem).

We now turn to describing the firms' optimization problem.

1.4 Firms' optimization problem

Any firm's objective is to maximize profits from production at each point in time. The firm takes as exogenous the level of rents it has to pay the production factors (capital and labor) and chooses its levels of demand for labor $L^{\mathcal{D}}$, and, respectively capital $K^{\mathcal{D}}$ that maximize its output:

$$(L_{it}^{\mathcal{D}},K_{it}^{\mathcal{D}})=\underset{L_{it},K_{it}}{\arg\max}\;F_t(L_{it},K_{it})-K_{it}D_{Kt}-L_{it}D_{Lt}. \tag{6}$$

The firm decides on $L^{\mathcal{D}}_{it}$ and $K^{\mathcal{D}}_{it}$ at time t. At the time the decision is made, D_{Kt}, D_{Lt}, and θ_t are observable. The firm also has an expectation of the aggregate labor supplied to the first economic sector, which we denote by $\bar{L}^{e}_{it}$.

The firm's first-order conditions are derived from Equations (1) and (6):

$$\frac{L^{\mathcal{D}}_{it}}{K^{\mathcal{D}}_{it}} = \left((1-a)\frac{\theta_t \bar{L}^{e}_{it}}{D_{Lt}}\right)^{1/a} \tag{7}$$

$$D_{Kt} = a\theta_t \bar{L}^{e}_{it}\left(\frac{L^{\mathcal{D}}_{it}}{K^{\mathcal{D}}_{it}}\right)^{1-a}. \tag{8}$$

Note that our technology exhibits constant returns to scale. Consequently, wages and dividends paid to capital are driven up until the firms' profit is zero. This makes the firm ownership structure irrelevant.

1.5 Equilibrium

A rational expectations equilibrium comprises security prices P_{st}, $s \in S$, wages D_{Lt}, dividends paid to the capital asset D_{Kt}, and expected labor level dedicated to the first sector $\bar{L}^{e}_{it}$, such that at any time t:

(A) Demand and supply of labor in the first sector are equal:

$$\sum_{i\in\mathcal{I}} L^{\mathcal{D}}_{it} = \sum_{j\in\mathcal{J}} L^{\mathcal{S}}_{jt} \quad \text{for all } t.$$

(B) The expected aggregate labor dedicated to the first sector is realized:

$$\bar{L}_t = \bar{L}^{e}_{it} \quad \text{for all } i,t.$$

(C) Demand and supply of capital are equal, provided some capital is needed in the first sector (recall that no capital is required for the second economic sector):

$$\sum_{i\in\mathcal{I}} K^{\mathcal{D}}_{it} = 1 \quad \text{for all } t \text{ for which } \sum_{i\in\mathcal{I}} K^{\mathcal{D}}_{it} > 0.$$

(D) The security market clears:

$$\sum_{j\in\mathcal{J}} \pi_{jKt} = 1 \quad \text{for all } t$$

$$\sum_{j\in\mathcal{J}} \pi_{jst} = 0 \quad \text{for all } t, \text{and for all } s \neq K.$$

With only the conditions (A)–(D), Matsuyama (1991) shows that multiple equilibria may exist. To avoid multiple equilibria when this case is possible, we are making an assumption designed to pin down the equilibrium:

(E) When there exist multiple solutions $(P_{st}, D_{Lt}, D_{Kt}, \bar{L}_t)$ satisfying (A)–(D), a "social planner" selects the Pareto-optimal solution, that is, that equilibrium for which total consumption[9] is maximized:

$$(P_{st}, D_{Lt}, D_{Kt}, \bar{L}_t) = \arg\max \left\{ \bar{L}_t D_{Lt} + (1 - \bar{L}_t^e) Z + D_{Kt} \right\} \text{ for each } t.$$

We proceed to characterizing the equilibrium in two stages. The first stage characterizes the rents D_{Lt} and D_{Kt}, and an aggregate expected labor level dedicated to the first sector $\bar{L}_{it}^e$, so that conditions (A)–(D) are met. As we shall show, conditions (A)–(D) yield multiple static equilibria, and (E) offers a mechanism to select among them in a way that makes the dynamic equilibrium unique. The second stage characterizes security prices such that the security market clears.

We start by first exploring the possible static equilibria, that is, those triplets $(D_{Lt}, D_{Kt}, \bar{L}_{it}^e)$ satisfying conditions (A)–(D).

First, because the firms might scale the production arbitrarily, we may assume that they do so as long as there is capital to be raised to enable production. Because the total mass of available capital is one, condition (C), which states that the market for K should clear, is always met. Furthermore, condition (D) will be useful when we calculate security prices, but because securities do not influence the real economy in our model, we may assume that condition (D) is met. Thus, to characterize those equilibria for which conditions (A)–(D) are met, it is enough to focus on (A) and (B) only.

Using Lemma 1, we can now characterize the set of equilibria satisfying conditions (A)–(D). The following result is proved in the Appendix:

Lemma 2. At any time t the following three static equilibria satisfy conditions (A)–(D) above:

(1) An equilibrium in which all labor is working in the sector generating the alternative wage (and none in the first sector, i.e., $\bar{L}_t = 0$) always exists. In this equilibrium $D_{Kt} = 0$ and $0 < D_{Lt} = Z_t$.

(2) An equilibrium in which all labor is dedicated to the first sector (and none to the sector generating the alternative wage, i.e., $\bar{L}_t = 1$) exists if and only if $\theta_t \geq (1-a)^{-1} Z_t$. In this equilibrium, $D_{Kt} = a\theta_t$ and $D_{Lt} = (1-a)\theta_t$.

(3) A mixed equilibrium in which some but not all labor is dedicated to the first sector (i.e., $0 < \bar{L}_t < 1$) exists if and only if $\theta_t > (1-a)^{-1} Z_t$. In this equilibrium, $D_{Kt} = a\theta_t$ and $D_{Lt} = Z_t$.

[9] Total consumption $C(t)$ is equal to the sum of dividends paid to the capital asset and wages. Wages are paid by the capital-intensive firms (in proportion of $\bar{L}_t$) and by the less capital-intensive sector (in proportion of $1 - \bar{L}_t$).

It is useful to compare the existence of multiple static equilibria satisfying conditions (A)–(D) with the case of the unique equilibrium encountered in a single sector production economy. In the classical production economy with one productive sector (i.e., no alternative wage) and without any labor externalities, workers are employed in the productive sector, and total output θ_t gets divided proportionally between labor and capital. That is, in this baseline case, we have:

$$
\begin{aligned}
D_{Kt}^{baseline} &= a\theta_t \\
D_{Lt}^{baseline} &= (1-a)\theta_t, \ \forall t.
\end{aligned} \tag{9}
$$

After characterizing those equilibria satisfying conditions (A)–(D), we can now turn to finding those that additionally satisfy (E). To do so, we first observe that at any time t for which $\theta_t < (1-a)^{-1} Z_t$, only the equilibrium in which all labor is employed in the second, less capital-intensive production is feasible, and thus this is also the Pareto-dominating equilibrium. For those times t when $\theta_t \geq (1-a)^{-1} Z_t$, however, all three equilibria described in Lemma 2 are feasible. Criterion (E) selects that particular equilibrium in which the rents paid out to labor and capital combined are maximized. The smallest total rent is paid out in the equilibrium in which all labor works in the less capital-intensive production technology; this total rent is equal to Z_t. The next smallest rent is paid out when the partial equilibrium is played; in this case, labor and capital combined receive $\bar{L}_t a\theta_t + (1-\bar{L}_t)Z_t$. Because in this case $\bar{L}_t < 1$ (that is, not all labor works in the first sector) and $\theta_t \geq (1-a)^{-1} Z_t$ (a condition for the partial equilibrium to exist), we observe that the total rent paid in the partial equilibrium is smaller than θ_t. However, θ_t is the total rent paid out in the equilibrium in which all labor is dedicated to the first sector. Thus, when $\theta_t \geq (1-a)^{-1} Z_t$, the Pareto-dominating equilibrium is the equilibrium in which all laborers are employed in the first sector. We can thus completely characterize the equilibria satisfying (A)–(E) as follows. As long as $\theta_t \geq (1-a)^{-1} Z_t$, all labor is employed in the first sector. The rent on capital is $D_{Kt} = a\theta_t$, and the rent on labor is $D_{Lt} = (1-a)\theta_t$. When $\theta_t \leq (1-a)^{-1} Z_t$, all labor is employed in the no-capital sector. The rent on capital is $D_{Kt} = 0$, and the rent on labor is the alternative wage, $D_{Lt} = Z_t$. Therefore, the nature of the equilibrium is described by the position of the state variable θ relative to the barrier $(1-a)^{-1} Z_t$.

Note, however, that the assumption of a social planner forcing the economy into the Pareto-optimal equilibrium might be too optimistic relative to what occurs in practice. For example, Murphy, Shleifer, and Vishny (1989) argued that an economy in a state of collapse does not recover as soon as this is feasible, but that it needs a "big push" to revert to the nondisaster state. In our model, this would imply that the economy, once collapsed, has to wait until θ_t climbs to a value strictly higher than $(1-a)^{-1} Z_t$ before it exits the crisis state. Accepting that a collapsed economy needs a big push to recover leads to longer crisis durations, and such a model would in turn exhibit higher volatility and higher equity premia than does our model. Formally, a big push equilibrium is obtained when assumption (E) is replaced with:

(E') An economy in which all labor is employed in the productive sector always selects the Pareto optimal equilibrium when multiple static equilibria are feasible. An economy in which labor works in the less capital-intensive sector must remain in that state as long as $\theta_t < \theta_t^g$, where θ_t^g is a positive threshold such that $\theta_t^g > (1-a)Z_t$. As soon as $\theta_t \geq \theta_t^g$, the economy selects the Pareto-optimal equilibrium.

Having detailed our equilibrium,[10] we turn to modeling choices for the productivity θ.

1.6 Productivity models and dynamic equilibrium

The variables determining the dynamics of our model are the productivity θ_t and the reservation wage Z_t. We model Z deterministically as

$$Z_t = Ze^{\mu t}, \tag{10}$$

where the rate of growth is $\mu > 0$. We model the productivity as an exponential function of a mean-reverting stochastic process, that is,

$$\theta_t = \theta_{0t} e^{X_t}, \tag{11}$$

$$dX_t = k(\overline{X} - X_t)dt + \sigma dW_t.$$

Above, θ_{0t} is a deterministic normalizing factor defined as:[11]

$$\theta_{0t} = (1-a)^{-1} Z_t. \tag{12}$$

With this specification, when $k > 0$, the productivity growth has a fixed standard deviation and a drift that mean reverts around a fixed value,[12] and Doob (1942) ensures that asset pricing quantities of interest, as functions of a stationary random variable, are also stationary. When the speed of mean reversion is $k = 0$, productivity is a lognormal Brownian motion with drift, similar to Nelson and Plosser (1982). As we shall show later, lognormal productivity results in normal

[10] We note that the model could be expanded to have partial unemployment as a possible equilibrium. This would be the case if different laborers had different alternative wages. As productivity drops, the wages could drop below the alternative wages for the first group (the group with the highest reservation wages). This would cause this group of workers to shift to the alternative production sector. The search cost will then go up, causing a further decline in aggregate output of the first sector. Wages will in turn drop (although labor share of output will rise). If these lower wages are higher than the alternative wages of the second group, the spiraling will stop, and we will have a partial unemployment. If these lower wages are below the alternative wages of the second group, the downward spiral will continue. For the spiraling to stop, the difference in the reservation wages needs to be relatively large, and/or the first group should be relatively small. As our intent is not to capture the political science aspects of economic and financial crises, and by no means do we try to explain unemployment, we keep the setting relatively simple, with only one alternative wage level and shifts from full employment to complete unemployment (and vice versa) that result from this rather simplified setting.

[11] The normalizing factor θ_{0t} is chosen so that $\theta_t \geq (1-a)^{-1} Z_t$ if and only if $X_t \geq 0$.

[12] The productivity has a volatility equal to σ and the drift of productivity growth mean-reverts around the fixed rate of $\mu + \sigma^2/2$.

returns for the capital asset in the baseline, one-sector case (but not in the case of a two-sector economy).

With the description of the modeling choice of θ completed, we can now characterize our dynamic equilibrium in terms of the state variable X.

Proposition 1. There exists a unique equilibrium satisfying conditions (A)–(E). Employment and rents are characterized as follows:

- If $X_t \geq 0$, all labor is employed in the first economic sector. The rent on capital is $D_{Kt} = a\theta_t$, and the rent on labor $D_{Lt} = (1-a)\theta_t$. Total consumption is equal to θ_t.
- If $X_t < 0$, all labor is employed in the less capital-intensive sector. The rent on capital is $D_{Kt} = 0$, and the rent on labor is $D_{Lt} = Z_t$. Total consumption is equal to Z_t.

From the proposition above, we readily observe that our economy exhibits a peso characteristic: as long as $X_t \geq 0$, labor and capital meet in a productive, capital-intensive economy and the consumption good is produced. Total consumption is $D_{Kt} + D_{Lt} = \theta_t = (1-a)^{-1} Z_t e^{X_t}$. When X_t is close to zero but positive, consumption gets close to $(1-a)^{-1} Z_t$, which is strictly greater than Z_t as a is between zero and one. As soon as X_t becomes negative, low productivity implies that wages in the capital-intensive sector decline sufficiently for labor to shift to the less capital-intensive sector. Total consumption–now fueled solely by wages–becomes Z_t. Thus, as X_t becomes negative, total consumption experiences a decline of $(1-a)^{-1} Z_t - Z_t$, that is, a drop equal to the rents on capital that are no longer paid. Therefore, consistent with the peso literature, X_t becoming negative[13] triggers a disaster, or a peso event. However, in contrast with previous work, we generate this disaster without modeling jumps in our state variable.

We continue by calculating risk premia and volatilities in our setting.

1.7 Security prices

In this subsection, we compute closed-form security prices of any security s at any time t using the Euler equation:

$$P_{st} u'(C_t) = \mathbb{E}_t \left[\int_t^{\infty} e^{-\delta(l-t)} u'(C_l) D_{sl} dl \right], \tag{13}$$

where C_t is the equilibrium consumption level of the representative agent at time t. Alternatively, we shall employ the notation $P_{st} = P_s(t, X_t)$ to emphasize dependence of the state variable X_t underlying our system's dynamics.

13 Note that to accommodate the big push equilibrium, Proposition 1 can be readily modified: in this case, our economy, once in a state of disaster, will remain in that state until the state variable X_t reaches a constant threshold X^g.This corresponds to the barrier $\theta_t^g = \theta_{0t} e^{X^g}$.

The calculation in Equation (13) is complicated by the fact that the stream of cash flows D_{st} depends on the state of the economy, and more specifically on whether labor works in the first sector or makes the alternative wage. However, if we know the state of the economy the functional form of dividends is known (and observable by everyone), and we shall exploit this computational convenience.

We continue by calculating the price of the risky asset and its conditional risk premium.

1.7.1 The price of the risky asset. For notational simplicity, when we calculate the price of the capital asset we shall drop the subscript K from the expression of the price; that is, we will denote $P_{Kt} = P_K(t, X_t)$ by P_t or $P(t, X_t)$.

To calculate the price of the capital asset, we proceed as follows. First, from the Euler equation observe that the prices P_s of any security s at the current time t and prices at any future time $t' > t$ are linked by:

$$P_s(t, X_t)u'(C_t) = \mathbb{E}_t\left[\int_t^{t'} e^{-\delta(l-t)}u'(C_l)D_{sl}dl + e^{-\delta(t'-t)}u'(C_{t'})P_s(t', X_{t'})\right]. \tag{14}$$

The formula above can be applied by taking the times t and t' to be those when the labor transitions between the capital-intensive sector and the sector ensuring the alternative wage.

For mathematical convenience, it is easier to calculate prices consistent not with the social planner equilibrium (in which the economy recovers as long as it is feasible to do so) but rather with the big push case. In this equilibrium, the economy does not recover as soon as $X \geq 0$ but instead only when $X_t \geq X^g > 0$. The social planner equilibrium prices can then be derived from the big push prices by taking the limit as $X^g \downarrow 0$.

We can now turn to describing the dynamic equilibrium implied by Proposition 1. Before we calculate prices for any values of the state variable X, it is helpful to start by calculating prices at the times when the economy shifts into and out of a crisis. Assume that P_0^b is the price of the risky asset the first time the economy collapses after time zero (i.e., at the first time that $X_t < 0$, denoted here by T_0). Once the economy collapses, it will take a time T_1 to recover (T_1 is how long it takes X to climb from 0 to X^g). At recovery, the price is P_0^g. Once the economy recovers, it will take some time, T_2, until the economy collapses again (this is how long it takes X to become negative, after starting at $X^g > 0$). The time between the two disasters is $T_1 + T_2$, and because prices are linear in the reservation wage Z_t,[14] when the economy collapses the

[14] This is because dividends on the capital asset, as well as consumption, are linear functions of Z_t. Because Z_t grows at a deterministic rate, for any security s and any times $0 \leq t < t'$ we have that $P_s(t, X_t = x) = e^{\mu(t'-t)} P_s(t', X_{t'} = x)$.

second time after zero, the price at that collapse will be $P^b_{T_0+T_1+T_2} = e^{\mu(T_1+T_2)} P^b_0$. After T_3 years elapse, the economy will recover for the second time (as X reaches X^g), and at that time the price will be $P^g_{T_0+T_1+T_2+T_3} = e^{\mu(T_2+T_3)} P^g_0$, and so on. Therefore, to calculate the prices at a collapse and at a recovery, we need to compute these values only for the first times the economy collapses and recovers. To do so, we apply formula (14) twice: once by taking $t := T_0$ (the first time of collapse) and $t' := T_0 + T_1$ (the time of the first recovery after the first collapse), and again by taking $t := T_0 + T_1$ (the time of the first recovery), and $t' := T_0 + T_1 + T_2$ (the time of the second collapse, which follows the first recovery). From these two relationships we can solve for P^b_0 and P^g_0, and thus find the prices at any recovery or collapse. We continue by calculating prices for the social planner case, that is, the case of $X^g \downarrow 0$. More specifically, the following is proven in the Appendix:

Proposition 2. Let $\ell(\cdot;\alpha)$ the function defined in Lemma A.1 and $g(\cdot;\alpha,\beta)$ the function defined in Lemma A.2. Then the price of the risky asset at a time t when the economy collapses, denoted by P^b_t, is described by:

$$P^b_t = \begin{cases} \dfrac{aZ_t}{(1-a)^{1-\gamma}} \; \dfrac{g'(0) - [\ell'(0) + \gamma - 1] g(0)}{\ell'(0)} & \text{if } k > 0 \\[2ex] \dfrac{aZ_t}{(1-a)^{1-\gamma}} \; \dfrac{1 - \dfrac{\sigma(\gamma-1)}{\sqrt{2\delta + 2(\gamma-1)\mu}}}{2\delta + 2(\gamma-1)\mu - \sigma^2(\gamma-1)^2} & \text{if } k = 0. \end{cases} \tag{15}$$

For brevity, we wrote $g(x) := g(x;\delta + (\gamma-1)\mu, \gamma-1)$ and $\ell(x) := \ell(x;\delta + (\gamma-1)\mu)$.

Applying formula (14) again, for any time t and for $t' = T_0$ (and noting that right before the economy collapses at time T_0, $P_{T_0} = P^g_0$), we can readily obtain the price of the capital asset for those times t when labor works in the capital-intensive sector. Furthermore, applying the pricing formula for any time t when labor works in the no-capital sector and for $t' = T_0 + T_1$ when the economy recovers, we can also obtain the price in the state of the economy in which all labor is dedicated to the second, less capital-intensive sector. The following is proved in the Appendix:

Proposition 3. Let P^b_t defined in Proposition 2, $\ell(\cdot;\alpha)$ the function described in Lemma A.1 and $g(\cdot;\alpha,\beta)$ the function described in Lemma A.2 and Equation (A1).

Conditional on labor being employed in the capital-intensive sector at time t, the following are true:

(1) The price of the risky asset is:

$$P(t,X_t)=\begin{cases} \left[\dfrac{aZe^{\mu t}}{1-a}g(0;\delta+(\gamma-1)\mu,\gamma-1)+(1-a)^{-\gamma}P_t^b\right]\times \\ \quad \times e^{\gamma X_t}\ell(X_t;\delta+(\gamma-1)\mu) \\ \quad -\dfrac{aZe^{\mu t}}{1-a}e^{X_t}g(X_t;\delta+(\gamma-1)\mu,\gamma-1) & \text{if } k>0 \\ \dfrac{aZe^{\mu t}}{1-a}\dfrac{e^{X_t}-e^{\left(\gamma-\frac{\sqrt{2\delta+2(\gamma-1)\mu}}{\sigma}\right)X_t}}{\delta+(\gamma-1)\mu-\sigma^2(\gamma-1)^2/2} \\ \quad +(1-a)^{-\gamma}P_t^b e^{\left(\gamma-\frac{\sqrt{2\delta+2(\gamma-1)\mu}}{\sigma}\right)X_t} & \text{if } k=0 \end{cases} \tag{16}$$

(2) Let $D(t,X_t)$ be the rate of dividends paid to the risky asset.[15] The volatility of the returns of the risky asset is given by:[16]

$$Vol(X_t)=\begin{cases} \sigma\gamma+\dfrac{e^{\gamma X_t}\left[\dfrac{aZ}{1-a}g(0)+\dfrac{P_0^b}{(1-a)^{\gamma}}\right]\ell'(X_t)-e^{X_t}g'(X_t)-(1-\gamma)e^{X_t}g(X_t)}{\left[\dfrac{aZ}{1-a}g(0)+\dfrac{P_0^b}{(1-a)^{\gamma}}\right]e^{\gamma X_t}\ell(X_t)-e^{X_t}g(X_t)} & \text{if } k>0 \\ (\sigma\gamma-\sqrt{2\delta+2(\gamma-1)\mu})+\dfrac{2}{\sigma(\gamma-1)+\sqrt{2\delta+2(\gamma-1)\mu}}\dfrac{D(t,X_t)}{P(t,X_t)} & \text{if } k=0 \end{cases} \tag{17}$$

(3) The expected excess return of the risky asset is given by:

$$\sigma\gamma\ Vol(X_t)\ \text{ for all } k\geq 0. \tag{18}$$

(4) The Sharpe ratio of the returns of the risky asset is constant and equal to $\sigma\gamma$.

Conditional on labor working in the less capital-intensive sector at time t, the following hold:

[15] From Proposition 1, $D(t,X_t)=a\theta_t=\dfrac{aZe^{\mu t}}{1-a}e^{X_t}$ when labor is employed in the capital intensive sector.

[16] For brevity, we wrote $g(x):=g(x;\delta+(\gamma-1)\mu,\gamma-1)$ and $\ell(x):=\ell(x;\delta+(\gamma-1)\mu)$.

(1) The price of the risky asset is:

$$P(t,X_t)=\begin{cases} P_t^b\ \ell(X_t) & \text{if } k>0 \\ P_t^b e^{\frac{\sqrt{2\delta+2(\gamma-1)\mu}}{\sigma}X_t} & \text{if } k=0. \end{cases} \tag{19}$$

(2) The volatility of the risky asset is:

$$Vol(X_t)=\begin{cases} \sigma\dfrac{\ell'(X_t)}{\ell(X_t)} & \text{if } k>0 \\ \sqrt{2\delta+2(\gamma-1)\mu} & \text{if } k=0. \end{cases} \tag{20}$$

(3) The expected excess return of the risky asset is equal to zero.

(4) The Sharpe ratio is equal to zero.

It is useful to compare these formulae with those obtained in the baseline case. Recall that in this case the economy features only one sector, with a constant intensity of capital a, and lognormal productivity (i.e., $k=0$). Absent the mechanism that triggers the collapse, the dividends to the capital asset are perpetually:

$$D^{baseline}(t,X_t)=\frac{aZe^{\mu t}}{1-a}e^{X_t},\ \text{for any } t>0. \tag{21}$$

In a similar vein, we could verify that the price of the risky asset satisfies:

$$P^{baseline}(t,X_t)=\frac{aZe^{\mu t}}{1-a}\frac{1}{\left[\delta+(\gamma-1)\mu-\frac{1}{2}\sigma^2(\gamma-1)^2\right]}e^{X_t}. \tag{22}$$

Furthermore, it follows that in this case the volatility of the capital asset is equal to $Vol^{baseline}(t,X_t)=\sigma$ (thus equal to the volatility of dividends) and that the risk premium is given by $\sigma^2\gamma$ (and therefore small). The baseline model, therefore, is *identical* to the canonical asset pricing model, and as such suffers from both the volatility and the equity premium puzzles. Furthermore, the equity premium and the volatility are constant, and cannot conditionally depend on the state of the economy.

We continue now by deriving empirical implications of our model and by showing that the asset prices we derived are consistent with a variety of asset pricing stylized facts.

2. Empirical Predictions

Having solved for prices of securities, we now turn to the predictions generated by our model. In the first section, we investigate the validity of our labor mechanism. We then demonstrate that our parsimonious model delivers many asset pricing stylized facts in a unified setting.

2.1 Capital's share of income and disasters

From Proposition 1, we can infer the size of a consumption drop in disasters. Specifically, a disaster translates into a decline of $Z_t/(1-a)-Z_t$ in aggregate consumption. In relative terms, this represents a $[Z_t/(1-a)-Z_t]/[Z_t/(1-a)] = a$ drop in total consumption during a disaster. Therefore, our model has a very simple testable implication:

Proposition 4. More capital intensive economies experience larger disasters.

We will test this prediction in our empirical section, and find support for it.

2.2 Volatility

In this section, we analyze the conditional volatilities produced by our model. The next results can be easily (but tediously) derived from formula (17). The complete proofs are in the Appendix. We start first with the case of no mean reversion in consumption growth, that is, the case of $k=0$.

Proposition 5. For $k=0$, conditional on the economy not being in a disaster state:

(1) The volatility of returns is higher than the volatility of dividends for any degree of relative risk aversion.
(2) The volatility is a nonincreasing function of prices.
(3) The volatility is an affine function of dividend yields.

We observe that for the case of no mean reversion in consumption, the model explains several stylized facts regarding volatilities. Specifically, our model predicts that prices are more volatile than dividends, thereby addressing the excess volatility puzzle (see Shiller 1981; Le Roy and Porter 1981; West 1988). Furthermore, the endogenous volatility generated by the model decreases with prices, consistent with empirical observations reported in the ARCH literature. This explains both the persistence observed in volatility and the asymmetric property of it. Because both prices and volatility are endogenous in our model, we are not proposing a volatility feedback mechanism (Campbell and Hentschel 1992; Bekaert and Wu 2000), in which an anticipated increase in volatility leads to a price decline. This result also should not be interpreted as a leverage effect (Black 1976; Christie 1982), in which lower prices drive the increased volatility; this effect should be expected even with no leverage in the capital structure. It is also worth mentioning that our model contrasts with Barro (2006) or Rietz (1988): whereas in these peso models the volatility of consumption equals the volatility of prices, in our model the latter is several times higher. Wachter (2013) generates a similar effect by modeling probabilities of disaster that are time varying, as ours also are. Changing the probability that a disaster occurs may change the frequency of the Rietz-Barro drops in prices and thus generate extra volatility.[17] In contrast with Wachter's model, ours has only a single factor and agents with CRRA preferences.

It has been noted that volatility is very persistent–French, Schwert, and Stambaugh (1987) note that autocorrelation of volatility remains high even after twelve monthly lags, and conclude that volatility is not stationary. We show that in our model, when the state variable is a random walk, the volatility is a function of this random walk and therefore is persistent. When the state variable is mean reverting, we show that volatility has more autocorrelation than the state variable. These implications of our model are consistent with the empirical observations on volatility stationarity.

Perhaps the most important implication that our model has about volatility is that the instantaneous volatility of the capital asset is itself a function of the state variable. Consequently, when the state variable indicates that a disaster is nearer, the representative agent requires more compensation for holding the risky asset. As in our model the Sharpe ratio is constant, when the disaster is more imminent, the volatility increases. By contrast, a Rietz-Barro disaster model generates volatility through having many drops in prices. In this type of exogenous crash model, disasters do occur without "warning signals" for investors. Therefore, absent a variable whose role is to describe how the distribution of collapses changes, a model with exogenous disasters cannot match the conditional moments of asset prices.[18]

Volatility is known to be higher in recessions. As recessions are periods in which equity prices are low, and because the volatility is a decreasing function of price when there is low mean reversion in consumption growth, our model can explain this stylized fact.

Volatility has been noted to react differently to a positive return innovation as compared with a negative return innovation: volatility tends to decrease after a realization of positive returns and increase after a realization of negative returns (Nelson 1991). A positive realization of returns in our model is the result of a positive innovation to the state variable.

From Equation (17), we can show that an increase in the state variable, within a normal range of values for that variable, results in reduced volatility. Likewise, a negative realization of returns in our model is the result of a negative innovation in the state variable. A decline in the state variable will thus result in increased volatility. These results are illustrated in Figure 1.

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17 Specifically, the lower the value of the state variable X_t (and thus the lower the consumption), the higher the probability of disaster when $k=0$.

18 Such a variable would furthermore be an additional state variable, and therefore, a model with exogenous disasters attempting to match conditional moments of asset prices will have at least two state variables. This is in contrast with our model, which can capture asset pricing dynamics with a single state variable.

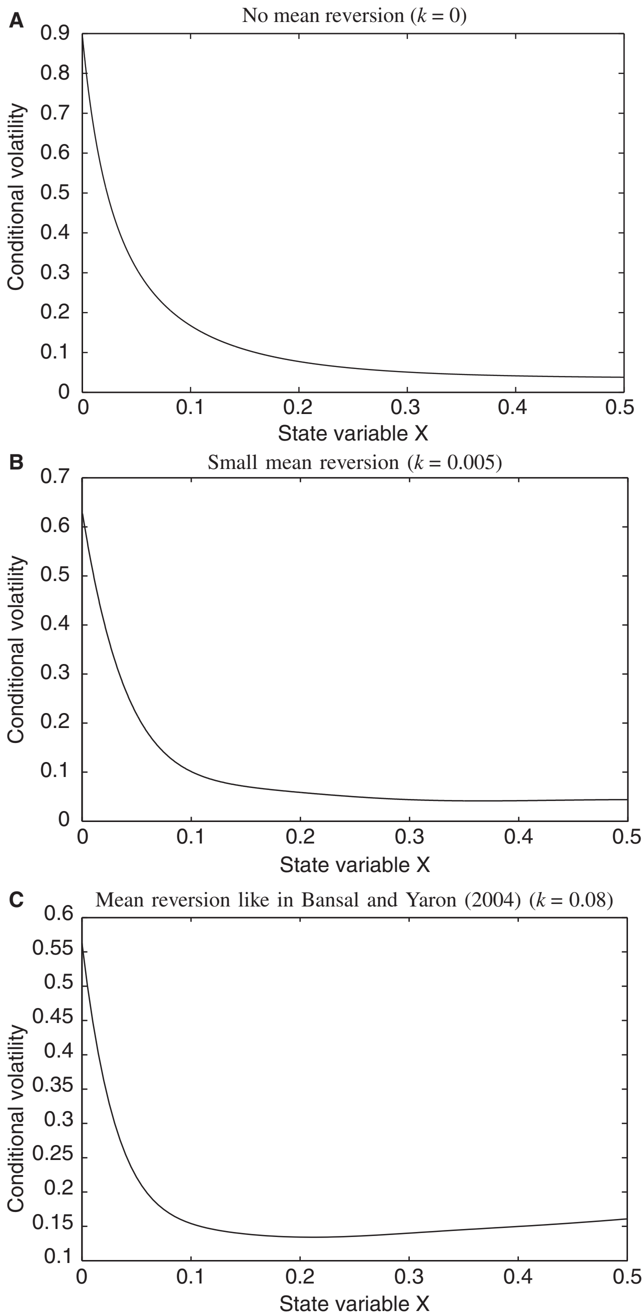

Figure 1
Conditional volatility as a function of the state variable
This figure presents the conditional volatility as a function of the state variable, conditional on no disasters. We present results for the three models we calibrate.

2.3 Expected returns

In this section, we show how our model is consistent with several empirical observations on the expected excess returns of the risky asset.

Proposition 6. Conditional on the economy not being in the disaster state, and when $k=0$:

(1) Expected excess returns are a decreasing function of prices.
(2) Expected excess returns are an affine function of dividend yields.

The model predicts that for no mean reversion in consumption growth, both expected excess returns and volatility are linearly related to dividend yields. Fama and French (1988) and Campbell and Shiller (1988) report that conditional excess returns are predicted by linear regressions with dividend yield as the explanatory variable. The model here gives a theoretical justification to these findings. Furthermore, as the model is based on rational agents, this result does not imply any market inefficiency. The result is not driven by any change in the agent's risk aversion as, for example, in habit formation models (Campbell and Cochrane 1999). Risk in the model moves in tandem with dividend yields: when the risk is higher, the agent requires a higher premium to hold the asset. Thus, dividend yields predict returns because they are positively related to expected returns.

Although in the general case of a mean-reverting state variable (i.e., $k>0$) volatility and expected excess returns are not affine functions of dividend yields, as they are both functions of the state variable, as are the dividend yields, it follows that expected excess returns, as well as volatilities, are also (nonlinear) functions of dividend yields. However, the functional form of this relationship is not monotonic when the speed of mean reversion in consumption growth is greater than zero. We plot the functional form of these relationships in Figure 2, for $k=0, k=0.005$, and $k=0.08$. While the pattern for $k=0$ was analyzed above, the patterns for $k>0$ (presented in Panels B and C of Figure 2) reveal another fact about predictability: in bad times (which would be characterized by times with high dividend-to-price ratios), there is an almost linear dependence of the risk premium on dividend yields. This would justify predictability of returns by dividend yields in bad times just as it does for the case $k=0$. By contrast, in good times (i.e., times when dividend yields are low), the dependence of expected excess returns on dividend yields is flat. This would imply in turn that in good times, dividend yields do not predict returns nearly as well as in bad times. This conditional predictability effect has been documented by Henkel, Martin, and Nardari (2011), and in a theoretical setting in which agents learn about regular and unusual fundamental states, David and Veronesi (Forthcoming) generate similar conditional patterns of volatilities as functions of price/earnings ratios.

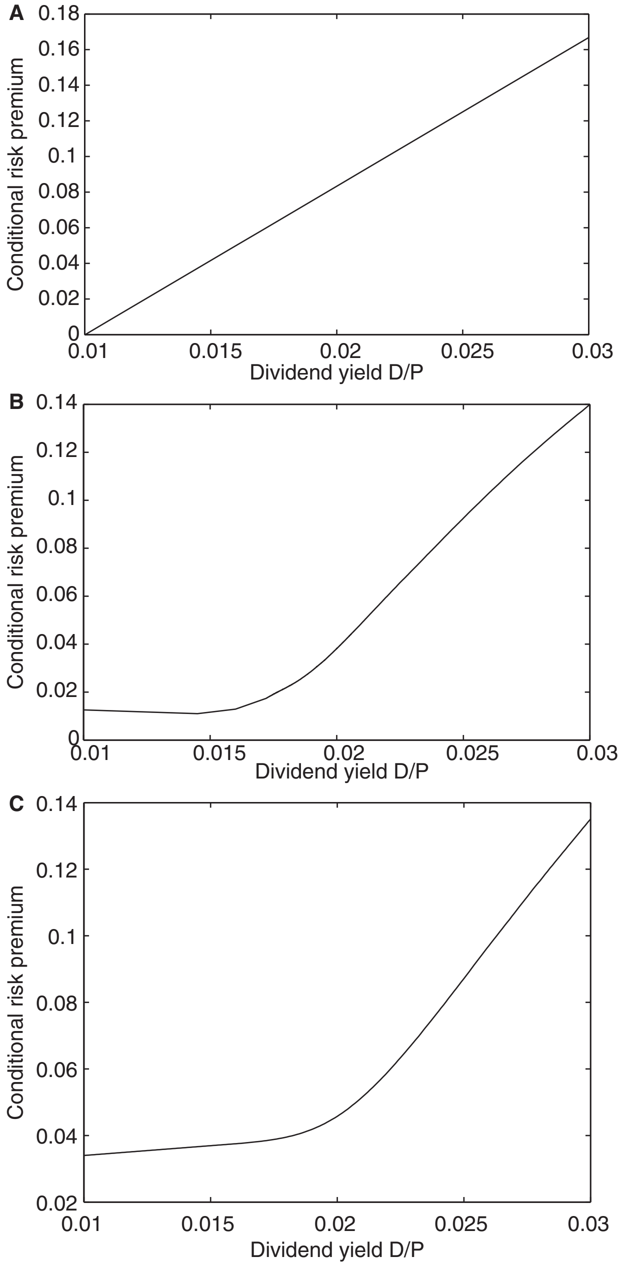

Figure 2
Dividend yields and the conditional contemporaneous equity premium
This figure presents the relationship between the dividend yield D/P and the conditional risk premium, that is, the drift of returns less the risk-free rate. We present results for the three models we calibrate.

3. Calibration and Empirical Tests

In this section we calibrate our model, test its empirical implications, and compare our results to empirical studies that tested similar predictions.

We present three distinct calibrations. In one calibration, log productivity is a random walk. In such a model, log consumption has a constant drift and volatility when labor is employed in the productive sector. Conditional on labor being fully employed in the capital intensive sector, therefore, consumption growth is i.i.d. (and therefore stationary). Furthermore, conditional on labor being employed in the less capital intensive sector, log consumption is deterministic, with a constant drift. In particular, without considering the less capital intensive sector, with this calibration our model reduces to the standard asset price paradigm of pricing a log-normal consumption stream. However, the transition between the cases in which all labor works in the capital intensive sector and the alternative case of all labor being employed in the less capital intensive sector is determined by the state variable X being positive or negative. Because X is a Brownian motion (and therefore nonstationary), it then results that log consumption is in turn nonstationary. As an alternative to nonstationary returns, we also offer two alternative calibrations in which the state variable is a mean-reverting diffusion and discuss their qualitative implications.

We proceed to describing the calibration.

3.1 Parameter calibration

We use the monthly returns of the value-weighted CRSP index for the period 1927–2011, the risk-free rate of the Federal Reserve's Publication H.15, and the Bureau of Labor Statistics' Consumer Price Index and productivity series.

The parameters of the model are $\Psi=(\delta,\mu,k,\overline{X},\sigma,\gamma,a,X_0)$. Note that the constant $Z>0$ is not among the parameters we need to fit: the reason is that from Equation (16) it can be inferred that prices are linear functions of Z. Together with the fact that dividends (to both labor and the risky asset) are also linear functions of Z and that the risk-free rate is independent of Z, this fact implies that expected returns, volatilities, and Sharpe ratios are independent of Z as well.

We start by selecting parameters that tackle the equity premium puzzle as it was exposed by Mehra and Prescott (1985). Thus, we select their proposed value for the volatility of consumption, conditional on the economy not being in a disaster state. As in these states of the world the consumption process is given by $\theta_t=\theta_{0t}e^{X_t}$, its volatility is equal to σ. Following Mehra and Prescott (1985), we thus use $\sigma=3.57\%$. We note that this may be an oversimplification, as the volatility of the time series of consumption varied over time and, in particular, decreased following World War II. Consumption of nondurable goods is the least volatile: Mehra and Prescott (1985) find its volatility to be 3.57% for their entire sample period, and according to Chapman (2002), it may be as low as

1% for the postwar period.[19] Alternatively, Barro (2006) and Wachter (2013) calibrate this volatility to 2%.[20] Furthermore, the consumption growth drift is mean reverting in our model. To select a value, we note that Nordhaus (2005) reports average consumption growth rates calculated in different periods and using different methods. For the United States, the rates he reports range from 1.24% to 2.53% per annum. We thus fix this long-run mean of consumption growth to 2%. Finally, the last parameter of interest for consumption growth is its speed of mean reversion. We use three distinct values for the speeds of mean reversion. First, we calibrate a model with a speed of mean reversion equal to zero. In this model the returns are not stationary, but we study it to demonstrate that our results are not driven by autocorrelation in consumption growth. We then present a calibration in which consumption growth in good times is mean reverting, with a speed of mean reversion inspired by Bansal and Yaron (2004). Bansal and Yaron (2004) report an autocorrelation in quarterly consumption growth that in our model would correspond to a speed of mean reversion of $k=0.08$, and therefore we present results for a model with this speed of mean reversion. Finally, we present a calibration in which the state variable is still stationary (mean reverting), albeit with a very small mean reversion rate set to $k=0.005$. Both the low mean reversion and the zero mean reversion calibrations appear to match well with qualitative aspects of the data.

We select risk aversion so that the Sharpe ratio implied by our model, conditional on the economy not being in a disaster state, is equal to 0.30. This amounts to selecting γ such that $\sigma\gamma=0.30$. We thus select $\gamma=8.40$. Note that Mehra and Prescott (1985) consider that a model with $\gamma<10$, with the observed volatility of consumption and the observed average equity premium, offers a resolution to the equity premium puzzle. Therefore, our model tackles the equity premium puzzle in the sense defined by Mehra and Prescott (1985).

We then set the value of a to correspond to the labor share of $1-a=0.64$ observed in the U.S. economy.[21] With this choice, as outlined in Section 1.6, the relative drop of consumption in a crisis will be equal to $a=36\%$. This is a larger drop than the 31% reported by Barro (2006) for the Great Depression. However, we find it supportive of our model that the consumption drop suggested by our setting is in range of what has been observed in the Great Depression.[22]

[19] Aït-Sahalia, Parker, and Yogo (2004) argue that consumption of luxury goods is what should matter, because equity holders are typically rich and satiated with the consumption of basic goods. They find that the volatility of luxury goods consumption is an order of a magnitude higher than the volatility of overall consumption.

[20] In our model, the volatility of dividends equals volatility of consumption in our model. Similar to consumption volatility, the volatility of dividends also varies in time. For example, it exceeds 10% for the entire available series in the United States, whereas it is smaller at around 5% in the postwar period.

[21] See Kydland and Prescott (1982).

[22] We note that this is also a much milder drop than if we calibrate the model assuming that the second sector was a primitive economy. If that is the case, then the production function in the second economic sector should only include raw labor, whose share of income is much lower. For example, in 1996, the raw labor's share of income was 4.9%, according to data compiled by Krueger (1999). This implies a drop of consumption in a disaster that is equal to 100−4.9% = 95.1%. Calibrating to such an extreme outcome is beyond the scope of this study.

Additionally, we set the value of the discount rate δ in a standard fashion. The savings literature, for example, Hubbard, Skinner, and Zeldes (1995), uses values consistent with a discount rate $e^{-\delta}=0.97$. Following this, we set $\delta=3.1\%$.

Finally, we fit the remaining parameters, which are the long-run mean of the state variable $\bar{X}$ and its initial value X_0. For the long-run mean of X, we note that we are able to map a relationship between its value and how long X spends in a disaster state (i.e., how long X stays under zero). We select $\bar{X}$ so that the average (simulated) time spent under zero matches the average duration of consumption disaster periods reported by Barro and Ursua (2008), which is 3.6 years. This yields values of $\bar{X}$ that are equal to 0.5793 when $k=0.005$ and to 0.0146 when $k=0.08$. Finally, in the case of no mean reversion (i.e., $k=0$), we do not need to fit $\bar{X}$ because it has no effect on the dynamics of the state variable X.

We then calculate the values of the state variable that are implied by the observed path of the U.S. economy. To do so, we first select X_0 such that the first moments of the Brownian innovations $W_{t+dt}-W_t$, as implied by real prices, correspond to a normal distribution $\mathcal{N}(0,dt)$ as the model assumes. To obtain these innovations, we use the time series of real returns on the risky asset. Specifically, we observe that by setting X_0, we can calculate the price $P(0,X_0)$ at the start of the calibration period. Using the returns in the data Ret_1 for the first time period, we may thus infer the price $P(1,X_1)=P(0,X_0)e^{Ret_1}$ at time 1. Using the formula for prices as given by Proposition 3, we can thus infer X_1, and so on. Once we have the time series of X, we could infer the Brownian increments as $W_{t+dt}-W_t=[X_{t+dt}-X_t-k(\bar{X}-X_t)dt]/\sigma$. When there is no mean reversion, the state variable starts at $X_0=0.1131$. With a slow speed of mean reversion, the initial value of the state variable is 0.0804, whereas in the case of mean reversion speed consistent with Bansal and Yaron (2004), the initial value of X is 0.0733.

Calibrating the model to price data follows a trend in disaster models (e.g., Barro 2006; Gourio 2008a; Wachter 2013), but this is not a unique choice. For example, Balvers and Huang (2007) calibrate a productivity-based model to productivity data, whereas Barro and Ursua (2008) show that these models may be calibrated to consumption data. Finally, in our model a disaster occurs immediately: consumption drops suddenly and stays low until recovery. Juilliard and Ghosh (2012) argue that it is important whether or not a disaster model is calibrated to match the cumulative consumption drop. Because the observed disasters were never instantaneous, it is important to calibrate a model consistent with an observed path of the economy in which such instantaneous drops were not observed in the U.S. data. It will become apparent that in two of our calibrations, the United States did not experience a disaster state.

A summary list of the parameters resulting from our three distinct calibrations (as differentiated through their respective values for the consumption growth mean reversion speed k) is presented in Table 1.

Table 1
Parameters resulting from calibration

Symbol	Definition	Values		
		Model 1	Model 2	Model 3
k	Speed of mean reversion for consumption growth	0.000	0.005	0.080
$\bar{X}$	Average distance from disaster	–	0.5793	0.0146
X_0	Initial value of the state variable	0.1131	0.0804	0.0733
σ	Volatility of consumption conditional on no disasters	3.57%	3.57%	3.57%
δ	Discount rate	3.1%	3.1%	3.1%
μ	Average consumption growth conditional on no disasters, $-\sigma^2/2$	2%	2%	2%
a	Capital's share of income	0.36	0.36	0.36
γ	Relative risk aversion	8.40	8.40	8.40

This table presents the model parameters resulting from various calibration procedures. The state variable X is mean reverting, following $dX_t = k(\bar{X} - X_t)dt + \sigma dW_t$.

3.2 Capital's share of income and disasters

In this subsection we will test our labor mechanism. To do so, we note that Proposition 4 assesses that more capital intensive economies should experience larger consumption drops in a disaster. This is precisely the implication we will test in this subsection.

To test this implication, we obtain data on labor's share of income from Gollin (2002) and on size of economic disasters from Barro and Ursua (2008). Intersecting the data presented in these two studies produces a small sample of twelve countries for which both labor's share of income and size of disasters are available.

Figure 3 presents the labor's share of income (on the x-axis) plotted against observed magnitudes of disasters (on the y-axis). Gollin (2002) reports both naively calculated labor's shares and adjustments designed to include the operating surplus of private, unincorporated enterprises into the income share. Following his work, we plot the naive labor's share as well as the adjustments. Whereas Gollin (2002) reports three such adjustments, we only use the first two, as using the third one would further decrease our already small sample. Barro and Ursua (2008) report magnitudes of consumption and GDP during recessions across the world. We use four different measures of disaster magnitude. Specifically, to proxy for the magnitude of a disaster, we use both consumption and GDP declines observed during recessions. In selecting the timing of these declines, we use declines that occurred closest to the time when labor's share of income was measured as well as the largest observed drop in a country's available data. Using three measures of labor's share and four measures of disaster size yields the twelve plots of Figure 3.

From Figure 3, we observe that the relationship between labor's share of income and magnitude of disasters is negative for all measures used, supporting our model's empirical implication, while simultaneously attesting to its robustness.[23] Furthermore, our model appears reasonably calibrated to

[23] The power of these tests is, however, small given that the sample contains only twelve data points.

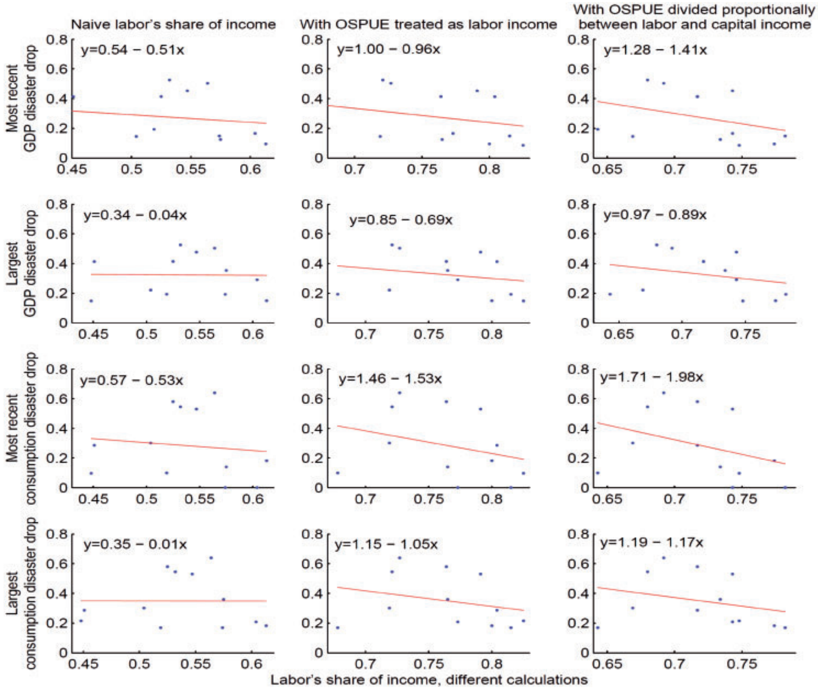

Figure 3
Labor's share of income and disaster magnitude
This figure presents the labor's share of income (on the x-axis) and the magnitude of consumption/GDP shocks during observed disasters for the following countries: Australia, Belgium, Finland, France, Italy, Japan, Netherlands, Norway, Portugal, Sweden, United Kingdom, and United States. The data are obtained from the intersection of Barro and Ursua (2008) (decrease in consumption/GDP) and from Gollin (2002) (labor's share of income). We report results for four measures of disaster magnitude (from Barro and Ursua 2008): GDP decline during a disaster (we use both the most recent observation as well as the largest decline) and consumption drop during a disaster (we use both the most recent observation as well as the largest decline). We use three measures of labor's share of income (from Gollin 2002): naively calculated labor's share of income, labor's share of income calculated adding the operating surplus of private unincorporated enterprises (OSPUE) to labor income, and labor's share of income calculated by proportionally dividing the operating surplus of private, unincorporated enterprises between labor and capital. A fitted linear relationship is reported for each plot.

address the size of observed disasters around the world. For example, the average disaster size among the twelve data points used in Figure 3 is 32.03%. This corresponds to a labor's share of income of $1-32.03\% = 67.97\%$, which is close to the value of 64% found in Kydland and Prescott (1982). The model also may be useful to shed light on the expected size of a disaster, which is equal to the capital's share of income. As Gollin (2002) puts the labor's share of income roughly between 65% and 80%, our model in turn implies a potential disaster size of 20% to 35%.

3.3 Conditional volatility and the equity premium

In this subsection we will discuss the empirical implications of our model as it was calibrated in subsection 3.1. While we argued that some stylized asset

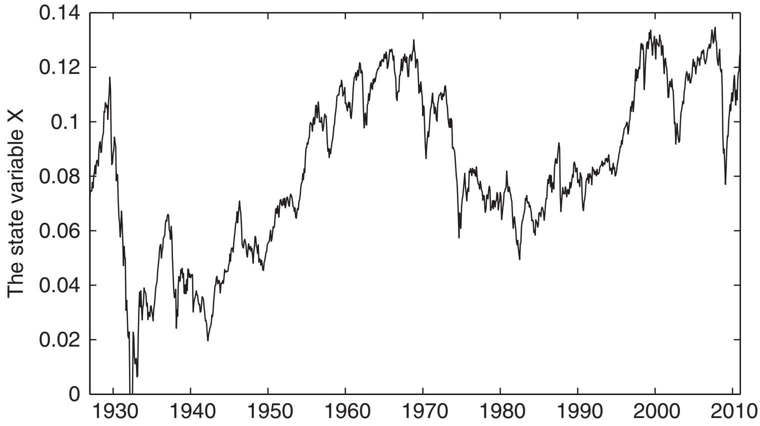

Figure 4
Conditional values for the state variable
This figure presents conditional values for the state variable X_t, for the no-mean reversion calibration ($k=0$).

pricing facts are matched theoretically by our model, in this section we further explore the quantitative insights offered by calibration.

We start by discussing how our model fits conditional moments of the observed equity premium in the U.S. economy. To obtain conditional values for the equity premium and for volatility, we simply have to apply the Equations (17) and (18) to the values of the state variable. An example of the values of the state variable for the case of $k=0$, is presented in Figure 4.

The conditional equity premia implied by our three calibrations are presented in Figure 5, whereas Panel A of Table 2 presents statistics on the conditional equity premium and on the volatility of the risky asset resulting from these calibrations. For example, the model calibrated using a rate of mean reversion consistent with Bansal and Yaron (2004) produces an average conditional risk premium of 5.61% and an average conditional volatility of 18.69%. The plots confirm the equity premium's countercyclicality, which seems to be more pronounced for the calibrations accounting for mean reversion in consumption growth. As apparent from the plots, the conditional equity premium ranges between 2% and 19%. These estimates seem consistent with previous estimates of the conditional ex ante equity premium (e.g., by Fama and French 2002 and Pastor and Stambaugh 2001).

We further simulate economies implied by our model[24] and calculate volatilities, risk premia and statistics on consumption and dividend growth along these simulated paths. Panel B of Table 2 presents results from these simulations. We observe that the average equity premia generated by our three

[24] To simulate paths for our economy, we could draw an initial value from the stable distribution of X, when X is mean reverting ($k>0$). For the case of $k=0$, however, we must make an assumption about X_0, and we choose the starting value of the U.S. observed economic path from a normal distribution centered at X_0.

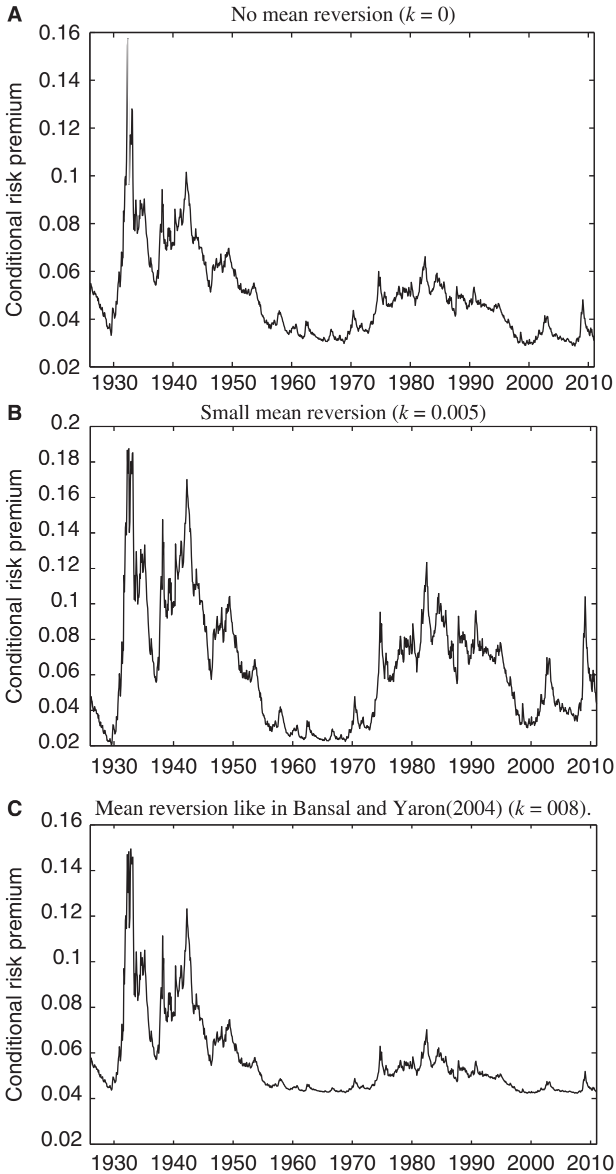

Figure 5
The conditional equity premium
This figure presents the conditional equity premium for the three models we calibrate.

Table 2
Calibration results

Panel A: Conditional moments

	Model 1	Model 2	Model 3
Av. eq. prem.	4.83%	6.34%	5.61%
Av. vol.	16.11%	21.13%	18.69%
Periods of disaster	yes	no	no
Panel B: Simulated moments, entire population			
Av. eq. prem.	6.01%	6.35%	6.28%
	(0.80%)	(1.02%)	(1.11%)
Av. vol.	20.02%	21.15%	20.93%
	(4.91%)	(7.32%)	(8.22%)
Av. cons. growth	2.18%	2.01%	2.08%
	(0.13%)	(0.22%)	(0.31%)
Av. cons. vol.	5.69%	4.93%	5.35%
	(1.21%)	(1.80%)	(2.84%)
Av. div. growth	1.98%	1.97%	1.99%
	(0.10%)	(0.03%)	(0.08%)
Av. div. vol.	10.08%	8.02%	11.40%
	(5.19%)	(3.73%)	(4.64%)
Av. risk-free ret.	11.52%	12.49%	7.67%
	(3.99%)	(0.79%)	(0.21%)
Av. risk-free vol.	0.01%	0.15%	2.40%
	(0.01%)	(0.02%)	(0.62%)
Panel C: Simulated moments, no disasters			
Av. eq. prem.	4.19%	3.61%	4.21%
	(1.83%)	(0.50%)	(0.25%)
Av. vol.	14.01%	12.02%	14.03%
	(2.90%)	(1.71%)	(0.67%)
Av. cons. growth	2.00%	2.01%	1.99%
	(0.03%)	(0.08%)	(0.11%)
Av. cons. vol.	3.57%	3.57%	3.57%
	(0.03%)	(0.04%)	(0.03%)
Av. div. growth	2.00%	2.01%	1.99%
	(0.11%)	(0.08%)	(0.11%)
Av. div. vol.	3.57%	3.57%	3.57%
	(0.02%)	(0.01%)	(0.02%)
Av. risk-free ret.	15.40%	13.83%	6.38%
	(0.00%)	(0.57%)	(0.37%)
Av. risk-free vol.	0.00%	0.15%	2.40%
	(0.00%)	(0.01%)	(0.05%)

This table presents alternative calibration results. Model 1 corresponds to the no–mean reversion case (that is, $k=0$). Model 2 corresponds to the case of small degree of mean reversion, and Model 3 assumes $k=0.08$ as in Bansal and Yaron (2004). Panel A presents averages for the conditional volatility and equity premium implied by our model for the observed path of the U.S. economy, between 01/1927 to 12/2011. If the state variable dropped below the disaster threshold during this observed sample path, we report that the U.S. economy experienced disasters. The disaster period reported in Model 1 extended from 02/1932 to 06/1932. Panels B and C report summary statistics of equity premium, volatility, and other asset pricing moments across 10,000 simulated paths of the U.S. economy, as well as the standard errors of these averages in parentheses.

calibrations compare well to that observed in our sample, which is 7.11% from January 1927 to December 2011. For example, the equity premia generated by a rate of mean reversion in consumption growth that is consistent with Bansal and Yaron (2004) is on average 6.28% (with a standard deviation across simulations of 1.11%). At the same time, the zero mean reversion calibration produces a lower average of 6.01%. The fact that the equity premium is still relatively high for the case in which no mean reversion is considered suggests that our results are not driven solely by the presence of long-run risks in consumption growth. The average simulated volatility is similar to that observed in our sample, which is 24.51% on an annualized level (we note, however, that our sample includes the volatile period of the Great Recession of 2008–2009). For example, in the case in which we allow for low mean reversion ($k=0.005$), the average simulated volatility is 21.15%, whereas this value is somewhat lower for the case without any mean reversion in consumption growth, at 20.02%. Another interesting fact made apparent by simulations is that our model exhibits an unconditional consumption growth volatility that is smaller than the volatility of dividend growth, which is consistent with empirical observations. This is because in our model, during a crisis dividends fall more than does consumption.[25]

It is important to note, however, the ex post returns in a peso environment may suffer from a positive survival bias as in Brown, Goetzmann, and Ross (1995), and we made certain to document that the magnitude of equity premium in our model is not driven by this bias. To do so, we separately analyzed the risk premiums and volatilities on those paths on which our economy did not suffer disasters. These results are presented in Panel C of Table 2. For the no-mean reversion calibration, for example, the average simulated equity premium for the paths which experienced no diaster during the 86 simulated years was 4.19%, with a standard deviation of 1.83%. This simulated value is less than two standard deviations away from the observed equity premium in the U.S. markets, which is 7.11% in our sample. The average volatility in the paths in which a disaster did not occur was 14.01%, with a standard deviation of 2.90%. By comparison, in a baseline model in which the economy has a single sector and no disasters occur, the volatility of prices would equal that of consumption, and would therefore be equal to $\sigma=3.57\%$. By contrast, our model has the capability to produce equity volatility that is about six times higher. These simulations show that the concept of a peso economy may offer a complete resolution to both the equity premium puzzle and the excess volatility puzzle. We note, however, that our model exhibits high risk-free rates, as we assume CRRA preferences for the representative agent. Barro (2006) circumvents this problem by arguing that the actual risk-free rate should be the risk-free rate derived in our model multiplied by an average recovery rate of treasuries that may default during disasters. Making such an assumption would decrease the risk-free rate while increasing the risk premium. Even though the level of the risk-free rate is high, it is apparent from Table 2 that its volatility is low, consistent with what has been observed empirically.

The comparison between the calibration with the speed of mean reversion used by Bansal and Yaron (2004) of $k=0.08$ and a small speed of mean reversion of $k=0.005$ deserves some discussion. First, the average conditional value of

25 Credit and gratitude for pointing this out to us go to an anonymous referee. Traditional peso models assume leverage to obtain this effect; by contrast, we do not need this assumption.

the equity premium as implied by the calibration with a small mean reversion rate of $k=0.005$ is higher at 6.34% than the same value for the case of Bansal and Yaron's mean reversion speed of $k=0.08$. Furthermore, the countercyclicality of equity premium is more pronounced in the case of a small mean reversion speed, as apparent in Panel B of Figure 5. Finally, the range of the conditional equity premium implied by the calibration with a small speed of mean reversion, as apparent in the same Figure 5, is wider. In particular, the equity premium takes values around the low bound of 2% during the nonvolatile late 1960s and reaches values as high as 19%, whereas in the calibration with a mean reversion speed of $k=0.08$, the equity premium appears bound from below by a relatively high value of 4% during the late 1960s and never reaches values higher than 15%.

Finally, we note that in our Pareto-optimal equilibrium disasters occur only when there is no alternative and that no arbitrary shifts of the economy into the disaster state are possible. To our model, one may also add unpredictable consumption disasters in the manner inspired by the current disaster literature (e.g., Wachter 2013). Adding such arbitrary disasters will have the effect of increasing the volatility of the risky asset (and hence, the equity premium), without changing the conditional implications of our model.

We now turn to examining returns predictability.

3.4 Returns predictability

In this subsection, we test the hypothesis that the dividend yield implied by our model predicts conditional excess returns, consistent with empirical observations (e.g., Campbell and Shiller 1988; Cochrane 1992; Fama and French 1988; Keim and Stambaugh 1986; Stambaugh 1999 pointed out that dividend yields predict excess returns). First, we note that in our model, the conditional risk premium is a function of dividend yield.[26] For the case of no-mean reversion, for example, this function is in fact affine. As conditional excess returns are an affine function of dividend yields (as they are proportional to conditional volatility, which in turn is an affine function of dividend yields), it is then natural that dividend yields predict returns one period ahead via a linear relationship (a regression equation). The functional forms of these relationships between dividend yields and conditional risk premiums for all the cases we calibrate are illustrated in Figure 2.

As dividend yields and expected excess returns are linked in every path of our economy, we can test the returns' predictability, as implied by our model, in two ways. In the first of these tests, for all data points in our U.S. sample, we calculate the levels of dividend yields implied by our model. This is possible, as using our model we can infer the value of the state variable X at that point in time, and the value of the dividend yield as predicted by our model is a function

[26] As implied by the third part of Proposition 5.

Table 3
In-sample return predictability, 1953–2011

	Model 1	Model 2	Model 3	Model 4	Model 5
Panel A: Quarterly predictability, 1953:Q1–2011:Q3					
Model implied D/P	0.15		0.17		0.26
	(2.48)		(2.71)		(1.00)
Actual D/P		0.27	−0.04		0.51
		(2.09)	(−0.15)		(0.29)
cay				0.81	1.03
				(2.86)	(2.60)
R^2	2.76%	1.53%	2.77%	3.62%	5.74%
Panel B: Annual predictability, 1953–2010					
Model implied D/P	0.19		0.31		0.37
	(3.23)		(2.33)		(2.20)
Actual D/P		0.25	0.03		0.01
		(3.01)	(0.12)		(0.11)
cay				1.13	0.38
				(3.18)	(1.14)
R^2	20.75%	25.67%	28.78%	16.27%	31.56%

This table presents predictive regressions of excess returns on dividend yields in the time series observed in the US economy between 1953:Q1 and 2011:Q3. We employ two separate variables for dividend yields: the first one is the dividend yield observed in the U.S. data. The second is the dividend yield as implied by our model, conditional on the observed equity returns in the U.S. *cay* is the Lettau and Ludvigson (2001) variable. t-stats are in parentheses. We use the calibration consistent with Bansal and Yaron (2004), in which the consumption growth mean reversion speed is $k = 0.08$.

of the state variable. This implied level of dividends will be different from the dividend yield observed in the data, for example, because distribution schemes used by firms in the real world consist of more than just paying dividends (see Boudoukh et al. 2007). By contrast, the dividends in our model are the only possible form of payout. We can then compare the predictability power of the dividend yields backed out from our model to that of the observed dividend yield or to other predictors of returns such as the *cay*, which has been proposed by Lettau and Ludvigson (2001). Because the data available on *cay* start in 1953, we restrict our sample to 1953:Q1 to 2011:Q3.

In Table 3 we report the results of such predictive regressions for 1953–2011. The frequencies reported are quarterly and annual, and, given the persistence shown by dividend yield ratios and documented by Lettau and Wachter (2007), results on predictability at shorter horizons should be interpreted with caution. In Table 3, we observe that the model-implied dividend yield predicts excess returns at quarterly and annual intervals and that its predictive power is higher than that of the observed dividend yield. At the longer (annual) horizon, the predictive power of the model-implied dividend yield subsumes that of *cay*. We can therefore conclude that our model-implied dividend yield passes this first predictability test.

To further assess the validity of our model, we verify that dividend yields predict excess returns at quarterly and annual horizons in simulated paths of our economy. Panel A of Table 4 presents averages of the coefficients of D/P, as well as averages of the R-squareds of these regressions. The numbers

Table 4
Simulated predictability

	$k=0.000$		$k=0.005$		$k=0.080$	
	β	R^2	β	R^2	β	R^2
Panel A: Returns predictability						
Quarterly	0.25	5.22%	0.22	3.48%	0.09	1.48%
Annually	0.39	11.21%	0.33	9.00%	0.18	6.44%
Panel B: Consumption growth predictability						
Quarterly	0.001	0.00%	0.008	0.17%	0.011	0.12%
Annually	0.001	0.01%	0.001	0.03%	0.011	0.05%

This table presents results from predictive regressions on simulated data from the three models we calibrate. For 10,000 simulated data samples, we aggregated simulated data quarterly and annually and we ran the regressions $R_{t+1:t+1+i} - rf_{t+1:t+1+i} = \alpha + \beta(D/P)_t + \epsilon_{t+i+1}$ (Panel A) or the regression $\Delta C_{t+1:t+1+i} = \alpha + \beta(D/P)_t + \epsilon_{t+i+1}$ (Panel B). Estimates of β and the R-squareds from both regressions are reported.

are consistent with stylized facts on returns, also confirmed by predictive regressions on U.S. data reported in Table 3. Finally, consumption growth is unpredictable in our model; we test whether dividend yields predict it and report the results of these tests in Panel B of Table 3. The results are consistent with the stylized fact that dividend yield does not predict consumption growth. This concludes our second battery of predictability tests.[27]

3.5 Time-series properties of Sharpe ratios

The ex ante Sharpe ratio in the model is constant, conditional on no disaster (see Proposition 3) and equal to $\sigma\gamma$. This might seem at odds with an observation that in the U.S. data, volatility and returns do not necessarily move together. Glosten, Jagannathan, and Runkle (1993), for example, find a negative relationship between risk and return. Whitelaw (1997) documents that Sharpe ratios vary considerably over time, and Whitelaw (2000) shows that in an economy with time-varying transition probabilities, the ex post time series of volatility and returns, under constant relative risk aversion (i.e., constant ex ante Sharpe ratio), exhibits a complex time-varying relation, which is negative in the long run. As the model we develop also has time-varying transition probabilities of disaster, we show similar properties of the ex post Sharpe ratio.

To study the time series properties of the realized Sharpe ratios in our model, we simulate daily paths of monthly histories, each one 1,032 months long. We draw paths of our model until we reach 10,000 paths that did not exhibit a disaster, consistent with the path observed in the U.S. economy. For each month, we compute the resulting volatility and the realized excess returns of that month. Dividing the two gives us the Sharpe ratio of that month. To document their time variability, we regress the 1,032 monthly Sharpe ratios on the dividend yield at the beginning of the month and record the statistical significance (i.e., the t-statistics) of this regression.

27 Note that we use the dividend yield D/P as a predictor, rather than $\log(D) - \log(P)$. This is due to the fact that, in our model, $D=0$ when the economy is in the disaster state.

The null hypothesis of constant Sharpe ratios would imply that they are not predictable by any variable. In that case, the t-statistics would be normally distributed with a zero mean. In the simulated model in which productivity is a random walk, however, this is not the case: the mean t-statistic is 2.87, and 81% of the simulated histories result in a statistically significant relationship between realized Sharpe ratios and dividend yield.

To understand what might drive this spurious predictability, it is instructive to analyze the variation in realized Sharpe ratios. A large negative shock at the beginning of the month (which can be predicted, as we have shown, by a low dividend yield at the end of the previous month) may cause volatility and expected returns to increase for the rest of the month. However, the change in expected returns is typically small, and most likely the end result for the month would be high volatility and negative returns. This effect weakens the relationship between realized returns and volatility, and makes realized Sharpe ratios seem to comove with returns (and hence to appear predictable by dividend yields). Another issue contributing to the spurious predictability of Sharpe ratios is the survival bias, which is driven by the fact that ex post, the possible break in the economy did not happen. With crises, a high dividend yield may be followed by either a period of recovery (and high returns) or by a crisis in which the risk premia and the Sharpe ratios are zero. Conditioning on disasters not occurring reduces the possibility that high dividends yields are followed by lower Sharpe ratios.

Finally, we note that the model could be extended to include sources of volatility that are orthogonal to our pricing kernel. For example, such volatility could arise if the division of output between labor and capital changed. This would affect asset prices, but because it does not change overall consumption, it would not command a price premium. The existence of such additional sources of volatility would cause Sharpe ratios to be even more countercyclical. Likewise, having σ decrease with X_t (i.e., a setting in which being farther away from the disaster triggers an increase in the stability of production) would result in even stronger countercyclical Sharpe ratios.[28]

4. Conclusions

Peso models solve many asset pricing puzzles and are testable. We add to this literature by proposing a parsimonious, one-factor model with CRRA agents in which the disasters occur endogenously. Having endogenous disasters not only conveys intuition about the link between crashes and the mechanism causing them but also allows agents the possibility to act as the disaster draws near. In our case, the nearer the disaster, the higher the market volatility.

[28] Such an assumption also has the potential to generate procyclical short-term rates. It is not surprising that disaster models have the potential to resolve a multitude of bond pricing puzzles. For example, Gabaix (2009) shows that different exogenous disaster specifications may solve a multitude of asset pricing puzzles. Bekaert, Hodrick, and Marshall (2001) also show that some term structure anomalies may be explained by peso arguments.

Our model produces the simple implication that more capital intensive economies experience larger disasters. It is also capable of matching several stylized asset pricing facts: because proximity to disasters attracts volatility, our model is capable of generating high volatility and, in turn, high equity premia, with a simple CRRA utility.

In addition, our model goes beyond these facts, in the sense that it is capable of addressing a series of *conditional* asset pricing stylized facts. For example, not only is the price volatility in the model high, but it is also countercyclical. Because both prices and volatility are endogenous, this is neither a volatility feedback effect nor is this a leverage effect, as in our model there is no leverage. The conditional values of the state variable appear to follow the state of the U.S. economy.

Because expected returns, as well as volatility of the risky asset, in our model are functions of dividend yields conditional on no disasters (and in a special case, affine functions), we offer a rationale for why dividend yields are excess returns predictors.

Whereas our model is similar to disaster models in which the probability of disaster is time varying, having only one factor and agents with constant relative risk aversion renders our simpler and more tractable.

To create our setting, we relied on existing economic models, which we extended by adding dynamics and then calculating asset prices. From this perspective none of the modeling assumptions are new, but their integration is. Given the minimality of our model, as well as its ability to match several asset pricing facts, we view our study as a step forward, moving the peso literature toward a unified asset pricing model.

Appendix

Proof of Lemma 2

If firms (rationally) expect that labor works in their sector, then $\bar{L}^e_{it}=1$ for each firm i. If also $D_{Lt}=(1-a)\theta_t$, then by Equation (7) it results that $K_{it}=L_{it}$ (for each firm). Because the aggregate supply of capital is one, the aggregate demand of labor will also be one, and the labor market will clear. This is also consistent with laborers being willing to supply labor to firms, as $D_{Lt}=(1-a)\theta_t > Z_t$.

When $\theta_t < (1-a)^{-1}Z_t$, labor working in the first sector is no longer feasible. Assume by absurd that firms expect labor will work for them, that is, $\bar{L}^e_{it}=1$. We clearly cannot have that $D_{Lt}\leq(1-a)\theta_t$, because this will result in $D_{Lt}<Z_t$, which means that the laborers will prefer to work in the less capital intensive sector, which pays the higher wage of Z_t. We must then have that $D_{Lt}>(1-a)\theta_t$. From Equation (7), it results that $L^{\mathcal{D}}_{it}<K^{\mathcal{D}}_{it}$ for each firm i. This however results in the supplied labor aggregating to $\sum_i L^{\mathcal{D}}_{it}<\sum_i K^{\mathcal{D}}_{it}=1$, which means that the labor market does not clear. This is a contradiction with the equilibrium definition.

Labor working in the less capital intensive sector is always possible: if each firm expects that all labor will be working in this sector, as clearly $D_{Lt}\geq Z_t>0$, from Equation (7) it results that $L^{\mathcal{D}}_{it}=0$ for each firm. With no labor, the firms' output is zero, and hence, the rents paid to capital are zero as well, that is, $D_{Kt}=0$. Both Equations (7) and (8) are satisfied regardless of the value of $K^{\mathcal{D}}_{it}$, in particular, we can select these values so that condition (C) of the equilibrium is met.

Finally, we address the case of mixed equilibrium. In the mixed equilibrium, the wages paid by the less capital intensive technology must be equal to the wages paid by capital intensive firms, that is, $D_{Lt}=Z_t$. From Equation (7), $L_{it}^{\mathcal{D}}=K_{it}^{\mathcal{D}}\left[(1-a)\theta_t \bar{L}_{it}^e Z_t^{-1}\right]^{1/a}$ for each firm i, and in order to sustain the condition (C) of the equilibrium as well as the condition (B), we must have that $\bar{L}_t=\left[(1-a)\theta_t \bar{L}_t Z_t^{-1}\right]^{1/a}$, or that $\bar{L}_t=[Z_t(1-a)^{-1}\theta_t^{-1}]^{1-a}$. Because in the mixed equilibrium case $\bar{L}_t<1$, this is sustainable if and only if $\theta_t>(1-a)^{-1}Z_t$.

We continue by presenting a few facts regarding functionals of Ornstein–Uhlenbeck processes, which are useful for our calculations. Some general sources of formulae used in this article are Borodin and Salminen (1996), Section 7 (for functionals involving the Ornstein–Uhlenbeck diffusions) and Abramowitz and Stegun (1964) (for the special functions involved in the calculations of the Ornestein–Uhlenbeck functionals, such as the definition of parabolic cylinder functions). ■

Functionals of Ornstein–Uhlenbeck processes

In what follows, we illustrate how to calculate in closed-form the functionals needed in our formulae. Unless otherwise specified, all the formulae below refer to an Ornstein–Uhlenbeck diffusion X, given by:

$$dX_t=k(\overline{X}-X_t)dt+\sigma dW_t.$$

The following lemma is from Borodin and Salminen (1996):

Lemma A.1. *Let $X_0=x$ and denote by T the first time that X reaches zero. Then for any $\alpha>0$,*

$$\ell(x;\alpha):=\mathbb{E}\left[e^{-\alpha T}|X_0=x\right]=\begin{cases}\dfrac{e^{k(x-\overline{X})^2/2\sigma^2}}{e^{k\overline{X}^2/2\sigma^2}}\dfrac{D_{-\alpha/k}(-(x-\overline{X})\sqrt{2k}/\sigma)}{D_{-\alpha/k}(\overline{X}\sqrt{2k}/\sigma)} & ,x<0\\[2ex] \dfrac{e^{k(x-\overline{X})^2/2\sigma^2}}{e^{k\overline{X}^2/2\sigma^2}}\dfrac{D_{-\alpha/k}((x-\overline{X})\sqrt{2k}/\sigma)}{D_{-\alpha/k}(-\overline{X}\sqrt{2k}/\sigma)} & ,x\geq 0\end{cases},$$

where D is the parabolic cylinder function.

Lemma A.2. *Let X_t be an Ornstein–Uhlenbeck process like above, and let T denote the first time X reaches zero. Then*

$$\mathbb{E}\left[\int_0^T e^{-\alpha t-\beta X_t}dt|X_0=x\right]=\ell(x;\alpha)g(0;\alpha,\beta)-e^{-\beta x}g(x;\alpha,\beta).$$

where $\ell(\cdot)$ is the function from Lemma A.2 and $g(\cdot;\alpha,\beta)$ is any (closed-form) solution of the Laplace equation:

$$g_{xx}+(\overline{b}-\overline{a}x)g_x+(\overline{d}-\overline{c}x)g=\overline{e}, \tag{A1}$$

with the constants $\overline{a},\overline{b},\overline{c},\overline{d},\overline{e}$ defined as:

$$\begin{aligned}\overline{a}&=2k/\sigma^2\\ \overline{b}&=-2\beta+2k\overline{X}/\sigma^2\\ \overline{c}&=-2k\beta/\sigma^2\\ \overline{d}&=\beta^2+2[-k\overline{X}\beta-\alpha]/\sigma^2\\ \overline{e}&=2/\sigma^2.\end{aligned} \tag{A2}$$

We continue by presenting a closed-form solution for $g(\cdot;\alpha,\beta)$ in Lemma A.2. For convenience we will drop the α,β from the script for g.

Closed-form solution for Equation (A1)

Equation (A1) is a version of the Laplace ordinary differential equation, and techniques to solve it in closed-form are described in Davies (1985, 342). Following Davies (1985), we start by looking for a solution of the form

$$g(x)=\int_{c_1}^{c_2} S(s)e^{sx}\,ds,$$

for some constants $c_{1,2}$. Substituting in (A1) and then integrating by parts, we are left with a first order ordinary differential equation for S, which can be solved in closed form. We select c_1, c_2 so that one of them is equal to $\pm\infty$ and the other constant is zero. The choice of $+\infty$ or $-\infty$ is such that the function $f(x)=ax+c$ does not have a root in the interval $[c_1, c_2]$. In our case, we select $c_1=-\infty,\ c_2=0$.

With this, we can readily check that the solution g is given by:

$$g(x)=\overline{e}|\overline{c}|^{\overline{d}+\overline{a}-\overline{b}\overline{c}/\overline{a}+\overline{c}^2/\overline{a}^2-1}\int_{-\infty}^{0}|\overline{a}s+\overline{c}|^{1-(\overline{c}+\overline{a}-\overline{b}\overline{c}/\overline{a}+\overline{c}^2/\overline{a}^2)}e^{-s^2/2\overline{a}-(\overline{b}/\overline{a}-\overline{c}/\overline{a}^2-x)s}\,ds. \tag{A3}$$

The derivative of g is given by:

$$g_x(x)=\overline{e}|\overline{c}|^{\overline{d}+\overline{a}-\overline{b}\overline{c}/\overline{a}+\overline{c}^2/\overline{a}^2-1}\int_{-\infty}^{0}|\overline{a}s+\overline{c}|^{1-(\overline{c}+\overline{a}-\overline{b}\overline{c}/\overline{a}+\overline{c}^2/\overline{a}^2)}se^{-s^2/2\overline{a}-(\overline{b}/\overline{a}-\overline{c}/\overline{a}^2-x)s}\,ds. \tag{A4}$$

The function g can be further simplified using the parabolic cylinder function D (see Abramowitz and Stegun, Ch. 19). To simplify notation, let $p:=1-(\overline{c}+\overline{a}-\overline{b}\overline{c}/\overline{a}+\overline{c}^2/\overline{a}^2)$, $q:=(2\overline{a})^{-1}$, and $r(x):=\overline{b}/\overline{a}-\overline{c}/\overline{a}^2-x$. Then

$$g(x)=\frac{e^{-\overline{c}^2q/\overline{a}^2+r(x)\overline{c}/\overline{a}}\,\overline{a}^{p-1/2}\sqrt{2}}{(2q)^{p/2}}\;\frac{2\pi D_{-p-1}\left(\sqrt{\overline{a}/2}(2\overline{c}q/\overline{a}^2-r(x)/\overline{a})\right)}{\Gamma(-p)e^{-(2\overline{c}-r(x))/(8\overline{a})}}.$$

Proof of Proposition 2

Before starting the proof, we start by outlining the idea. First, as we mention at the outset, our model is a limit of a big push-type model. In the big push model, the economy collapses as soon as the state variable X becomes negative, and it does not recover until X reaches a strictly positive value X^g. We shall thus start by deriving the pricing relationships at first for a general $X^g>0$, then take the limit as $X^g\to 0$. We further note that the values for the dividends paid to the risky asset and for the consumption process are known at any point in time.

In the general case, an economy started at $X_0=x>0$, in full employment, will remain in this state until $X_t<0$. When that happens, at time T_0, the price of the risky asset becomes P_0^b, and the economy shifts into the disaster state. The economy will remain in this state until time T_0+T_1, and at that time the economy will recover, and the price of the risky asset will become P_0^g.[29] Once recovered, the economy will remain in full employment until X becomes negative again, at time $T_0+T_1+T_2$. Despite the fact that the economy is in the same state at time $T_0+T_1+T_2$ as it was at time T_0, due to growth the price of the risky asset is now $e^{\mu(T_1+T_2)}P_0^b$. Similarly, at time $T_0+T_1+T_2+T_3$, when X will reach X^g and the economy will recover the second time, the price of the risky asset will be $e^{\mu(T_2+T_3)}P_0^g$. In general, as mentioned in Section 1.7.1, for any times $t<t'$, the price of the risky asset at the same value of the state variable X satisfies $P(t,X_t=x)=e^{\mu(t'-t)}P(t',X_{t'}=x)$. Therefore, in order to calculate prices when the economy collapses or recovers, we only have to calculate these prices at the first collapse and first recovery–any other values can be inferred by growing these prices at the rate μ. Finally, note that from Proposition 1, the price of the risky asset at zero is the price right *before* the economy collapses, in particular, that as $X_t\downarrow 0$ near the

[29] The subscript equal to zero is meant to signify that these are the first observed collapses and recoveries on a path started conventionally at time $t=0$.

first time of recovery T_0+T_1, then $\lim_{X_t\downarrow 0} P(t,X_t)=P_0^g$ (so the price as a function of the state variable X is continuous from the *right*); and that if t is around the first time of collapse T_0, $\lim_{X_t\uparrow 0} P(t,X_t)=P_0^b<P(T_0,0)$ (in other words, the function $P(t,\cdot)$ has a finite left limit).

We start by first applying the pricing Equation (14) for a time t where the state variable $X_t=0$, right before the time T_0 when the economy collapses, and for a time $t'>T_0$, for which the economy is already in the collapse state (such a time exists almost surely because the economy recovers only after X reaches $X^g>0$). Because no dividends are paid after the economy collapses, $u'(C_t)P(t,0)=\mathbb{E}[e^{-\delta(t'-t)}u'(C_{t'})P(t',X_{t'})]$. Taking the limit as $t'\downarrow 0$, we obtain that $P_0^b=(1-a)^\gamma P(t,0)$. Taking the limit as $X^g\downarrow 0$, we obtain that $P_0^b=(1-a)^\gamma P_0^g$. To obtain a second relationship linking P_0^b to P_0^g, it is useful to analyze the behavior of our economy between a recovery time $t:=T_0+T_1$ (where the price is P_0^g, and the state variable is equal to X^g), and the stopping time t' when the state variable (started at time T_0+T_1 from X^g) reaches zero (immediately afterward, the economy slips into the disaster state because there are no isolated zeros of a Brownian motion). Pushing at the same time $X^g\downarrow 0$, so at recovery, $X_t=0$, we obtain that:

$$u'(C_0)P_0^g=\mathbb{E}\left[\int_{T_0+T_1}^{t'} e^{-\delta(s-T_0-T_1)}u'(C_s)D_s\,ds\right]+\mathbb{E}\left[e^{-\delta(t')}u'(C_{t'})e^{\mu(t'-T_0-T_1)}P_0^g\right].$$

Above, the last term in the equation takes into account the fact that if X^g is close to zero, then the price at zero converges to $e^{\mu(t'-T_0-T_1)}P^g$. Substituting the expressions for consumption and dividends paid to the capital asset, we obtain that:

$$P_0^g=\frac{aZ}{1-a}\mathbb{E}\left[\int_{T_0+T_1}^{t'} e^{-[\delta+(\gamma-1)\mu]s-(\gamma-1)X_s}\,ds\right]+P_0^g\,\mathbb{E}\left[e^{-[\delta+(\gamma-1)\mu](t'-T_0-T_1)}\right].$$

Isolating P_0^g we obtain that:

$$P_0^g\left(1-\mathbb{E}[e^{-(\delta+(\gamma-1)\mu)t'}]\right)=\mathbb{E}\left[\int_{T_0+T_1}^{t'} e^{-[\delta+(\gamma-1)\mu]s-(\gamma-1)X_s}\,ds\right],$$

or equivalently, that

$$P_0^g=\frac{\mathbb{E}\left[\int_{T_0+T_1}^{t'} e^{-[\delta+(\gamma-1)\mu]s-(\gamma-1)X_s}\,ds\right]}{1-\mathbb{E}[e^{-(\delta+(\gamma-1)\mu)t'}]}.$$

We can then use Lemmas A.1 and A.2 to calculate the expectations above. As both the numerator and the denominator go to zero as $X^g\downarrow 0$, we can use l'Hospital's rule to find the limit and uncover the formula for P_0^g. We then use that $P_0^g=(1-a)^\gamma P_0^b$ to find P^b.

For the case when $k=0$, the calculations are simplified because some of the Brownian Motion functionals take simpler algebraic forms. Specifically, if $k=0$ and $X_t=\sigma W_t$ with W being a Brownian Motion, Borodin and Salminen (1996) gives that:

$$\mathbb{E}[e^{-[\delta+(\gamma-1)\mu]t'}]=e^{-X^g\sqrt{2\delta+2(\gamma-1)\mu)}/\sigma}.$$

Furthermore,

$$M_t:=\int_{T_0+T_1}^{t-T_0-T_1} e^{-[\delta+(\gamma-1)\mu]s-(\gamma-1)X_s}\,ds+\frac{e^{-[\delta+(\gamma-1)\mu]t-(\gamma-1)X_t}}{\delta+(\gamma-1)\mu-\frac{1}{2}\sigma^2(\gamma-1)^2}$$

is a martingale. Applying the optional sampling theorem, we obtain that $\mathbb{E}M_{t'}=M_0$. Noting that $X_{T_0+T_1}=\sigma W^g$ and that $M_0=1/\left(\delta+(\gamma-1)\mu-\sigma^2(\gamma-1)^2/2\right)$, we obtain a closed form expression for P^g. Using again l'Hospital's rule we obtain the formula in Proposition 2. ■

Proof of Proposition 3

We can then apply Equation (14) one more time with t being the current time and $t' = T_0$, the first time that X_t becomes negative, and we can thus obtain the equation for the price of the capital asset (16), conditional on all labor being employed in the capital intensive sector at time $t < T_0$:

$$u'(\theta_t)P(t,X_t)=\mathbb{E}_t\left[\int_t^{T_0} u'(\theta_l)e^{-\delta(l-t)}D_l dl + e^{-\delta(T_0-t)}u'(\theta_{T_0+})e^{\mu(T_0-t)}P_0^b\right].$$

In the above equation, D_l represents the dividend to the risky asset paid at time l if all labor is employed in the first sector. Thus $D_l = a\theta_l$. We can next use the formulae in Lemmas A.1 and A.2 to calculate the expectation of the integral above, as well as the expectation of the exponential of the hitting time of X, and we can readily obtain the formula for the price of the capital asset. For the particular case of $k=0$, we simply note that the formulae of Lemmas A.1 and A.2 simplify (using the same martingale argument as in the proof of Proposition 2). For example,

$$\mathbb{E}\left[\int_t^{T_0} e^{-[\delta+(\gamma-1)\mu](l-t)-(\gamma-1)X_l}dl\right]=\frac{e^{(1-\gamma)X_t}-\mathbb{E}[e^{-[\delta+(\gamma-1)\mu](T_0-t)}]}{\delta+(\gamma-1)\mu-\sigma^2(\gamma-1)^2/2},$$

and the expectation of the exponential stopping time can be further calculated as

$$\mathbb{E}[e^{-[\delta+(\gamma-1)\mu](T_0-t)}]=e^{-X_t\sqrt{2\delta+2(\gamma-1)\mu}/\sigma}.$$

This proves the first point of the Proposition.

To prove point 2, note that conditional on labor working fully in the first economic sector, and with the dividends to the capital asset being $D(t,X_t)=a\theta_t$, applying Itô's lemma to $P(t,X_t)$ implies that the returns R of the capital asset are given by:

$$dR_t=\frac{dP(t,W_t)+D(t,X_t)dt}{P(t,X_t)}=\left(\frac{P_t(t,X_t)+\frac{1}{2}\sigma^2 P_{XX}(t,X_t)+D(t,X_t)}{P(t,X_t)}\right)dt+\frac{P_X(t,X_t)}{P(t,X_t)}dX_t. \tag{A5}$$

As $dX_t = k(\overline{X}-X_t)dt+\sigma dW_t$, the volatility of the capital asset is given by:

$$Vol(t,X_t)=\sigma\frac{P_X(t,X_t)}{P(t,X_t)}.$$

A simple differentiation of the price in part 1 of the proposition produces the formula for the volatility of the capital asset for the case of $k>0$. For the case of $k=0$, the identity showing that the volatility $\sigma P_X/P$ is reduced to the formula in the Proposition can be verified readily.

The prices, as well as the volatility in the case when the economy is in the less capital intensive state, can be readily obtained as follows. First, note that in this state, rents to the capital asset are zero, and by Euler's pricing equation simplifies to

$$P(t,X_t)=\mathbb{E}\left[e^{-[\delta+(\gamma-1)\mu]T}\right](1-a)^{-\gamma}P_0^g,$$

where T is the time the economy spends in the disaster state (which is equal to the time it takes X to reach X^g, when the economy recovers, and then we push $^g \downarrow 0$). When $k=0$ the expression of the expectation above further simplifies to $e^{\sqrt{2\delta+2(\gamma-1)\mu}X_t/\sigma}$. This, coupled with the equation $P_0^g=(1-a)^\gamma P_0^b$, yields the expression for price. Volatility can be calculated, just as in the case in which the economy is in the capital intensive state, to be $\sigma P_X/P$; the expression for the volatility follows.

To calculate expected excess returns, we need to calculate first the risk-free rate. It is convenient to start by calculating the price of a riskree bond with maturity $\tau>0$. Let this price be $B(t,X_t,\tau)$.

In what follows, it will be useful to note that for each time τ, X_τ is normally distributed. Let

$$m_{t,\tau} := X_t + (\overline{X} - X_t)(1 - e^{-k\tau})$$

$$v_\tau := \frac{\sigma\sqrt{e^{2k\tau} - 1}}{\sqrt{2k}e^{k\tau}}.$$

Then, if X starts at X_t at time t, after some time τ elapsed, we have that

$$X_{t+\tau} \sim \mathcal{N}\left(m_{t,\tau}, v_\tau\right).$$

Let us first focus on the case in which the economy is in the capital intensive state. In this case,

$$u'(\theta_t)B(t, X_t, \tau) = e^{-\delta\tau}\mathbb{E}\left[u'(\theta_{t+\tau})1_{\{X_{t+\tau}\geq 0\}} + u'(Z_t e^{\tau\mu})1_{\{X_{t+\tau}<0\}}\right].$$

Substituting θ_t, we obtain that:

$$B(t, X_t, \tau) = e^{-\delta\tau - \gamma\mu + \gamma X_t}\mathbb{E}\left[e^{-\gamma X_{t+\tau}}\right] + \mathbb{E}\left[(1 - e^{-\gamma X_{t+\tau}})1_{\{X_{t+\tau}<0\}}\right].$$

Above, the first expectation is given by:

$$\mathbb{E}\left[e^{-\gamma X_{t+\tau}}\right] = \frac{1}{\sqrt{2\pi v_\tau}}\int_{-\infty}^{\infty} e^{-\gamma x} e^{-\dfrac{(x - m_{t,\tau})^2}{2v_\tau}} dx.$$

Completing the square under the integral, changing the variable so that we are left integrating the normal density function, we can simplify the above expression to:

$$\mathbb{E}\left[e^{-\gamma X_{t+\tau}}\right] = e^{-\gamma[X_t + \overline{X}(1 - e^{-k\tau})] + \frac{\gamma^2\sigma^2(e^{2k\tau} - 1)}{2ke^{2k\tau}}}.$$

If we are interested in calculating the risk-free rate, we are interested in the yield of the above bond when its maturity goes to zero. The yield of the bond B is given by $y(t, X_t, \tau) = -\log[B(t, X_t, \tau)]/\tau$. As $\tau \downarrow 0$, we note that $1_{\{X_{t+\tau}<0\}}$ goes to zero almost surely, and since it is bounded, the dominated convergence theorem implies that around $\tau = 0+$ we have that:

$$r_f(t, X_t) := y(t, X_t, 0+) = \lim_{\tau\downarrow 0}\frac{1}{\tau}\left[(\delta + \gamma\mu)\tau - \gamma X_t + \gamma X_t e^{-k\tau} + \gamma\overline{X}(1 - e^{-k\tau}) - \frac{\gamma^2\sigma^2(e^{2k\tau} - 1)}{4ke^{2k\tau}}\right].$$

It can be easily verified that the limit above results in a risk-free rate

$$r_f(t, X_t) = \delta + \gamma\mu - \gamma^2\sigma^2/2 + k\gamma(\overline{X} - X_t).$$

Note that the risk-free rate is large in our model (as it contains a term of $\gamma\mu$), as we have used a power utility. Two solutions exist to decrease the risk-free rate resulting from the equilibrium. The first was proposed by Barro (2006), who argues that the actual risk-free rate we use should be equal to r_f as implied by our calculations, multiplied by an average recovery rate reflecting the probability of default of treasuries during disasters. Making such an assumption would increase the equity premium of the model, leaving the volatility unchanged. The other method to decrease the risk-free rate would be to employ Epstein and Zin utility functions. In this case, we are unable to obtain closed-form solutions for asset prices and the risk free rate. Within our model, we can show that we can obtain a high volatility and a correspondingly high equity premium using the standard CRRA utility function.

In a similar vein, conditional on the economy being in the less capital intensive state,

$$r_f(t, X_t) = \delta + \gamma\mu.$$

We now turn to calculating the risk premium. We start first with the case in which the economy is in the capital intensive state at time t.

From formula (A5), we also calculate the expected returns on the risky asset as:

$$\mu(t,X_t) := \frac{P_t(t,X_t) + \frac{1}{2}P^{XX}(t,X_t) + D(t,X_t)}{P(t,X_t)}.$$

The partial derivatives above can be calculated, albeit tediously. First, $P_t(t,X_t) = \mu P(t,X_t)$. For the other partial derivatives, note that P is a function of parabolic cylinder functions $D_\nu(\cdot)$. To differentiate these functions, we make use of the fact that $D'_\nu(x) = -(x/2)D_\nu(x) + \nu D_{\nu-1}(x) = (x/2)D_\nu(x) - D_{\nu+1}(x)$.

For the case of $k=0$, the partial derivatives above are easier to calculate. After (in both cases tedious) algebra, it can be shown that

$$\mu(t,X_t) = r_f(t,X_t) + \sigma\gamma Vol(t,X_t).$$

In particular, this implies that the Sharpe ratio is constant, and equal to $\sigma\gamma$, conditional on the economy using capital intensive production.

The risk premium for the case in which the economy plays the less capital intensive equilibrium can be derived similarly.

Finally, although we only have one state variable, it is worth mentioning that our model produces a variety of term structure implications. Specifically, we can numerically show the following: (1) medium-maturity bonds have yields that are increasing in the state variable. This is "flight-to-quality" property observed in bonds; (2) term structure is U shaped. Such a structure has been observed in the United Kingdom (Brown and Schaefer 1994). A downward-sloping term structure (as we have at low maturities) has been documented by Evans (1998). ■

Proof of Proposition 5

There are two cases that we can consider. The first is the case of $\sigma\gamma - \sqrt{2\delta + 2(\gamma-1)\mu} \geq \sigma$. In this case, since D/P is positive, it results that $Vol(X) \geq \sigma$.

The second case is that of $\sigma\gamma - \sqrt{2\delta + 2(\gamma-1)\mu} < \sigma$. In this case, observe that D/P is decreasing in X and converges to $[2\sigma + 2(\gamma-1)\mu - \sigma^2(\gamma-1)^2]/2$ when $X \to \infty$. Therefore, the volatility is bounded from below by

$$\sigma\gamma - \sqrt{2\delta + 2(\gamma-1)\mu} + \frac{2}{\sigma(\gamma-1) + \sqrt{2\delta + 2(\gamma-1)\mu}} \frac{2\delta + 2(\gamma-1)\mu - \sigma^2(\gamma-1)^2}{2} = \sigma.$$

Point 2 follows, as dividend yields are decreasing as X increases, while at the same time prices increase with X. ■

Proof of Proposition 6

Since conditional on no disasters the equity premium is equal to $\sigma\gamma Vol(W_t)$ and from Proposition 5 we have that Vol is affine in dividend yields and nonincreasing in X_t, it results that the equity premium has the same properties.

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